Math 2552 Practice Problems for final

Question 1. Please answer the following short questions and show all your work for full credit.

(1) Find the longest interval in which the solution of the initial value problem

$$(t-0.5)\frac{dx}{dt} = tx + t^2 y$$
$$(\cos t)\frac{dy}{dt} = (\sin t)x + \frac{1}{t-2}y$$
$$x(1) = 1, \quad y(1) = 2$$

is certain to exist.

- This is a system of linear equations
- Write the system in standard form. $\frac{dx}{dt} = \frac{t}{t-0.5}x + \frac{t^2}{t-0.5}y.$

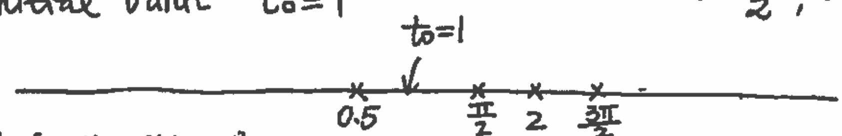
$$\frac{dy}{dt} = \frac{\sin t}{\cos t}x + \frac{1}{(\cos t)(t-2)}y.$$

- The coefficients are continuous except $t = 0.5,\ 2.$

$t = \frac{\pi}{2},\ \frac{3}{2}\pi,\ \frac{5}{2}\pi \cdots$

and $-\frac{\pi}{2},\ -\frac{3}{2}\pi,\ -\frac{5}{2}\pi, \cdots$

- initial value $t_0 = 1$



$$\boxed{\left(0.5, \frac{\pi}{2}\right)}$$

(2) Is the function $f(t) = e^{t^3}$ of exponential order? Why?

A function $f(t)$ is of exponential order if there exist $M, a, k \geq 0$ such that $|f(t)| \leq ke^{at}$, for $t \geq M$.

For $f(t) = e^{t^3}$, no matter what a is.

$$\lim_{t\to\infty} \frac{e^{t^3}}{e^{at}} = \lim_{t\to\infty} e^{t(t^2-a)} = +\infty.$$

which means $f(t) = e^{t^3}$ grows much faster than e^{at}.

So $f(t) = e^{t^3}$ is not of exponential order.

(3) For the non-homogeneous equation $y'' + 2y' + y = t^3 e^{-t}$, determine a suitable form for the particular solution if the method of underdetermined coefficients is to be used. Do not solve for the coefficients.

The characteristic eq. for the homogeneous eq. $y''+2y'+y=0$.

is $\lambda^2+2\lambda+1=0$ $\quad (\lambda+1)^2=0 \Rightarrow \lambda_1=-1 \quad \lambda_2=-1$

$$Y(t) = t^2(At^3+Bt^2+Ct+D)e^{-t}$$

$\uparrow$

because -1 appears twice in the roots. of the characteristic equation.

(4) Transform the following 4th order linear equation into a system of first order equations

$$y^{(4)} - ty''' + y'' = \sin t.$$

Let $x_1 = y$ $\qquad x_1' = y' = x_2$

$x_2 = y'$ $\qquad x_2' = y'' = x_3$

$x_3 = y''$ $\qquad x_3' = y''' = x_4$

$x_4 = y'''$ $\qquad x_4' = y^{(4)} = ty''' - y'' + \sin t$

$= -x_3 + t x_4 + \sin t.$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin t \end{bmatrix}$$

Question 2. Consider the differential equation

$$(1 + xy^3) + cx^2y^2\frac{dy}{dx} = 0.$$

(1) In order for the equation above to be exact, what should c be?

The equation is $\underbrace{(1+xy^3)}_{M}dx + \underbrace{cx^2y^2}_{N}dy = 0.$

It is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

$\frac{\partial M}{\partial y} = 3xy^2.$ let $3xy^2 = 2cxy^2.$

$\Downarrow$

$\frac{\partial N}{\partial x} = 2cxy^2$ $c = \frac{3}{2}.$

(2) Solve the equation with the c found in (1).

Let $c = \frac{3}{2}$

To solve this exact equation, we need to find $\phi(x,y)$

such that $\frac{\partial \phi}{\partial x} = M = 1+xy^3$

$\frac{\partial \phi}{\partial y} = N = \frac{3}{2}x^2y^2.$

$$\phi = \int M\,dx = \int (1+xy^3)\,dx = x + \frac{1}{2}x^2y^3 + h(y)$$

$$\frac{\partial \phi}{\partial y} = \frac{3}{2}x^2y^2 + h'(y) = \frac{3}{2}x^2y^2 \Rightarrow h'(y) = 0. \Rightarrow h(y) = C$$

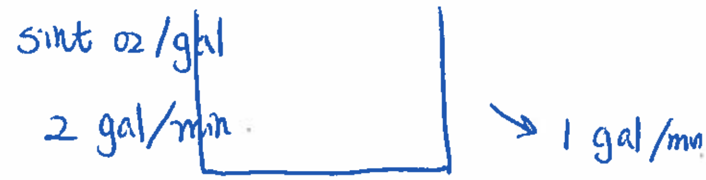
So: $\phi(x,y) = x + \frac{1}{2}x^2y^3 + C$

The general solution is $x + \frac{1}{2}x^2y^3 = C.$

Question 4. The tank initially contains 100 gal of fresh water. Water containing a salt concentration of $\sin t$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the rate of 1 gal/min. Find the amount of salt in the tank at any time.

Let $Q(t)$ be the amount of salt in the tank at time t.

$Q(0) = 0$.



$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} \qquad \text{Initial: } 100 \text{ gal water.}$$

$$= 2\sin t - \frac{Q}{100+2t-t}.$$

$$\frac{dQ}{dt} + \frac{Q}{100+t} = 2\sin t. \qquad \text{First order linear equation.}$$

Integrating factor $\mu(t) = e^{\int \frac{1}{t+100} dt} = e^{\ln(t+100)} = t+100.$

$$\mu(t)\frac{dQ}{dt} + \mu(t)\frac{Q}{t+100} = 2(t+100)\sin t.$$

$$\frac{d}{dt}\left[\mu(t) Q(t)\right] = 2(t+100)\sin t.$$

$$\mu(t)Q(t) = \int 2(t+100)\sin t \, dt + C.$$

$$= 2\int t\sin t \, dt + 200\int \sin t \, dt + C.$$

$$= 2(\sin t - t\cos t) - 200\cos t + C.$$

$$= 2\sin t - 200\cos t - 2t\cos t + C.$$

$$Q(t) = \frac{2\sin t - 200\cos t - 2t\cos t + C}{t+100}$$

$Q(0) = 0$. $\dfrac{-200 + C}{100} = 0 \Rightarrow C = 200.$ $Q(t) = \dfrac{2\sin t - 200\cos t - 2t\cos t + 2}{t+100}$

Question 5

(1) Show that $\mathbf{x}_1 = \begin{bmatrix} t \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} t^{\frac{1}{4}} \\ \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix}$ form a fundamental set of solutions for the homogeneous system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\mathbf{x}, \quad t>0.$$

You need to check that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, and they are linearly independent.

$$\frac{d\vec{x}_1}{dt} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\vec{x}_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{dx_2}{dt} = \begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} \\ -\frac{3}{16}t^{-\frac{7}{4}} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\begin{bmatrix} t^{1/4} \\ \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} \\ -\frac{3}{16}t^{-\frac{7}{4}} \end{bmatrix}$$

$$W[\vec{x}_1\ \vec{x}_2] = \begin{vmatrix} t & t^{\frac{1}{4}} \\ 1 & \frac{1}{4}t^{-\frac{3}{4}} \end{vmatrix} = \frac{1}{4}t^{\frac{1}{4}} - t^{\frac{1}{4}} = -\frac{3}{4}t^{\frac{1}{4}} \neq 0.$$

Yes, $\vec{x}_1$, $\vec{x}_2$ are solutions and they are linearly independent.

(2) Find the general solution of the following non-homogeneous system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{5}{4t} \end{bmatrix}.$$

Let $X(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} t & t^{1/4} \\ 1 & \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix}$ $\qquad \vec{g}(t) = \begin{bmatrix} 0 \\ \frac{5}{4t} \end{bmatrix}$

A particular sol. is $\vec{x}_p(t) = X(t)\vec{u}(t)$ where $\vec{u}' = X(t)^{-1}\vec{g}(t)$.

$$\det X(t) = -\frac{3}{4}t^{\frac{1}{4}} \qquad X(t)^{-1} = \frac{1}{-\frac{3}{4}t^{\frac{1}{4}}}\begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} & -t^{\frac{1}{4}} \\ -1 & t \end{bmatrix}$$

$$= -\frac{4}{3}t^{-\frac{1}{4}}\begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} & -t^{\frac{1}{4}} \\ -1 & t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}t^{-1} & \frac{4}{3} \\ \frac{4}{3}t^{-\frac{1}{4}} & -\frac{4}{3}t^{\frac{3}{4}} \end{bmatrix}$$

$$\vec{u}'(t) = \begin{bmatrix} -\frac{1}{3t} & \frac{4}{3} \\ \frac{4}{3}t^{-\frac{1}{4}} & -\frac{4}{3}t^{\frac{3}{4}} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{5}{4t} \end{bmatrix} = \begin{bmatrix} \frac{5}{3t} \\ -\frac{5}{3}t^{-\frac{1}{4}} \end{bmatrix} = \begin{bmatrix} \frac{5}{3}t^{-1} \\ -\frac{5}{3}t^{-\frac{1}{4}} \end{bmatrix}$$

$$\vec{u}(t) = \frac{5}{3}\begin{bmatrix} \ln t \\ 4t^{\frac{3}{4}} \end{bmatrix} \qquad \vec{x}_p = \begin{bmatrix} t & t^{1/4} \\ 1 & \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix}\begin{bmatrix} \frac{5}{3}\ln t \\ 5\ldots \end{bmatrix} = \begin{bmatrix} \frac{5}{3}t\ln t + \frac{5}{12}t \\ 5\ln t + \frac{5}{\ldots} \end{bmatrix}$$

General solution of the non-homogeneous system.

$$\vec{X}(t) = c_1 \begin{bmatrix} t \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t^{\frac{1}{4}} \\ \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix} + \begin{bmatrix} -\frac{5}{3}t\ln t + \frac{5}{12}t \\ -\frac{5}{3}\ln t + \frac{5}{48} \end{bmatrix}$$

Question 6 Consider the system of equations

$$\mathbf{x}' = \overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & a \end{bmatrix}} \mathbf{x}.$$

(1) If $a = 0$, what is the type and stability of the critical point $(0,0)$? Then find the solution to the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{bmatrix}$ $\quad \det(A-\lambda I) = -\lambda(1-\lambda) - 6$

$= \lambda^2 - \lambda - 6$

$= (\lambda-3)(\lambda+2)$

$\lambda_1 = 3$ $\quad \lambda_2 = -2$ $\quad \boxed{\text{saddle}}$

Eigenvector. $\lambda_1 = 3$ $\quad \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$ $\quad \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad 3v_1 + 2v_2 = 0$ $\quad \vec{v}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

General sol. $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{-2t}$ $\quad$ When $t=0$ $\quad c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$c_1 + 2c_2 = 4$

$c_1 - 3c_2 = -1$

$\Rightarrow 5c_2 = 5 \quad c_2 = 1$

$\Rightarrow c_1 = 2.$

Solution.

$\vec{x}(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-2t}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

(2) In order for $(0,0)$ to be a spiral sink, what interval should a be in? Show your work. If it can never be a spiral sink, explain why.

$A = \begin{bmatrix} 1 & 2 \\ 3 & a \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & a-\lambda \end{bmatrix}$

$\det(A-\lambda I) = (1-\lambda)(a-\lambda) - 6$

$= \lambda^2 - a\lambda - \lambda + a - 6.$

$= \lambda^2 - (a+1)\lambda + a - 6. = 0$

$$\lambda = \frac{a+1 \pm \sqrt{(a+1)^2 - 4(a-6)}}{2}$$

$$= \frac{a+1 \pm \sqrt{a^2+2a+1-4a+24}}{2} = \frac{a+1 \pm \sqrt{a^2-2a+25}}{2}.$$

In order to have a spiral sink (stable spiral)., we need

$a+1 < 0$ and $a^2 - 2a + 25 < 0.$

$a < -1$ and $a^2 - 2a + 25 = (a+1)^2 + 24 < 0$

which is impossible.

So. $(0,0)$ can never be a spiral sink.

Question 7. Compute the Laplace transform of the following functions:

(1) $f(t) = te^{-t}u_1(t)$.

(2) $f(t) = \begin{cases} 2 & 0 \le t < 4; \\ t & t \ge 4. \end{cases}$

(1). $\mathcal{L}\{u_1(t)\} = \frac{e^{-s}}{s}$.

$\mathcal{L}\{t u_1(t)\} = -\frac{d}{ds}\left(\frac{e^{-s}}{s}\right) = -\frac{(e^{-s})' s - e^{-s} \cdot 1}{s^2} = \frac{e^{-s} + e^{-s}s}{s^2}$.

$= e^{-s}\frac{s+1}{s^2}$.

$\mathcal{L}\{e^{-t} t u_1(t)\} = \mathcal{L}\{t u_1(t)\}\Big|_{s \to s+1} = e^{-(s+1)}\frac{s+2}{(s+1)^2}$.

(2). $f(t) = 2u_{0,4}(t) - t u_4(t)$

$= 2(1 - u_4(t)) - t u_4(t) = 2 - 2u_4(t) - t u_4(t)$.

$= 2 - (t+2) u_4(t) = 2 - (t - 4 + 6) u_4(t)$

$= 2 - (t-4) u_4(t) - 6 u_4(t)$

$\mathcal{L}\{u_4(t)\} = \frac{e^{-4s}}{s}$.

$\mathcal{L}\{\underbrace{(t-4)}_{f(t)=t} u_4(t)\} = e^{-4s}\mathcal{L}\{t\} = e^{-4s}\frac{1}{s^2}$.

$\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{e^{-4s}}{s^2} - \frac{6e^{-4s}}{s}$.

Question 8. Compute the Inverse Laplace transform of the following functions:

(1) $F(s) = \frac{2-s}{s^2+2s+2}$.

(2) $F(s) = \frac{e^{-3s}}{s(s^2+1)}$.

(1) $F(s) = \frac{2-s}{s^2+2s+1+1} = \frac{2-s}{(s+1)^2+1} = \frac{2-(s+1-1)}{(s+1)^2+1} = \frac{-(s+1)+3}{(s+1)^2+1}$

$= -\frac{s+1}{(s+1)^2+1} + 3\frac{1}{(s+1)^2+1}$

$\mathcal{L}^{-1}\{F\} = -\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$.

$= -e^{-t}\cos t + 3e^{-t}\sin t$.

(2). $\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s(s^2+1)}\right\} = u_3(t)\, g(t-3)$.

where $g = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$.

Partial fraction for $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

$1 = A(s^2+1) + s(Bs+C) = As^2 + A + Bs^2 + Cs$.

$= (A+B)s^2 + Cs + A$.

$A+B=0$, $C=0$, $A=1$ $\Rightarrow$ $A=1$, $B=-1$, $C=0$.

$\frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{-s}{s^2+1}$.

$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = \cancel{t}\ 1 - \cos t$.

Question 9. Use the Laplace transform to solve the following equation:

$$y^{(4)} - y = 0$$

with the initial value $y(0) = 1, y'(0) = 0, y''(0) = 1, y'''(0) = 0$.

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = 0.$$

Let $Y = \mathcal{L}\{y\}$.

$$s^4 Y - s^3 y(0) - s^2 y'(0) - s\, y''(0) - y'''(0) - Y = 0$$

$$s^4 Y - s^3 - s - Y = 0.$$

$$(s^4 - 1) Y = s^3 + s \quad \Rightarrow Y = \frac{s(s^2+1)}{(s^2+1)(s+1)(s-1)} = \frac{s}{(s+1)(s-1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s-1)}\right\}.$$

Partial fraction $\quad \frac{s}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$

$$s = A(s-1) + B(s+1).$$

Let $s=1 \quad 2B = 1 \quad B = \frac{1}{2}$

$s = -1 \quad -2A = -1 \quad A = \frac{1}{2}.$

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}.$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\Big|_{s\to s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\Big|_{s\to s-1}\right\}$$

$$= \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}.$$

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Problem [redacted]: Consider the initial value problem:

$$y'' + 4y = g(t), \quad y(0) = 0,\ y'(0) = 0$$

where

$$g(t) = \begin{cases} e^{2t} & \text{if } t \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

(1) Express $g(t)$ in terms of heaviside functions. (2 points)

$$g(t) = e^{2t} u_1(t).$$

(2) Compute $\mathcal{L}\{g\}$. (4 points)

$$\mathcal{L}\{e^{2t} u_1(t)\} = F(s-2) \quad \text{where } F(s) = \mathcal{L}\{u_1(t)\} = \frac{e^{-s}}{s}$$

$$= \frac{e^{-(s-2)}}{s-2} = \frac{e^{2}e^{-s}}{s-2}.$$

(3) Use the Laplace transform to solve the initial value problem. (12 points)

Apply the Laplace transform on this DE.

$$s^2 Y - s\,y(0) - y'(0) + 4Y = \mathcal{L}\{g\}$$

$$(s^2+4)\,Y = \frac{e^2 e^{-s}}{s-2}.$$

$$Y = \frac{e^2 e^{-s}}{(s-2)(s^2+4)}$$

$$y(t) = e^2 u_1(t) f(t-1) \quad \text{where } f = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s^2+4)}\right\}$$

$$\frac{1}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4}$$

$$1 = A(s^2+4) + (Bs+C)(s-2). = As^2 + 4A + Bs^2 - 2Bs + Cs - 2C$$
$$= (A+B)s^2 + (C-2B)s + 4A - 2C$$

Let $s=2$ $\quad 8A = 1 \quad A = \frac{1}{8}$

$A + B = 0 \quad B = -A = -\frac{1}{8}.$

$C = 2B = \frac{1}{4}$

$$\frac{1}{(s-2)(s^2+4)} = \frac{\frac{1}{8}}{s-2} + \frac{-\frac{1}{8}s}{s^2+4} + \frac{\frac{1}{4}}{s^2+4}$$

$$= \frac{1}{8}\frac{1}{s-2} - \frac{1}{8}\frac{s}{s^2+2^2} + \frac{1}{8}\frac{2}{s^2+2^2}.$$

Then $y(t) = \frac{1}{8}e^{2t} - \frac{1}{8}\cos 2t + \frac{1}{8}\sin 2t.$

Question 8. Find the general solution of the following system of equations

$$\mathbf{x}' = \underbrace{\begin{bmatrix} -7 & 9 & -6 \\ -8 & 11 & -7 \\ -2 & 3 & -1 \end{bmatrix}}_{A} \mathbf{x}.$$

$$A-\lambda I = \begin{bmatrix} -7-\lambda & 9 & -6 \\ -8 & 11-\lambda & -7 \\ -2 & 3 & -1-\lambda \end{bmatrix} -4③+②.$$

Elementary row operations $\rightarrow \begin{bmatrix} -7-\lambda & 9 & -6 \\ 0 & -1-\lambda & -3+4\lambda \\ -2 & 3 & -1-\lambda \end{bmatrix}$

$$\det A-\lambda I = (-7-\lambda)\begin{vmatrix} -1-\lambda & -3+4\lambda \\ 3 & -1-\lambda \end{vmatrix} -2 \begin{vmatrix} 9 & -6 \\ -1-\lambda & -3+4\lambda \end{vmatrix}.$$

$$= (-7-\lambda)\left[(\lambda+1)^2 - 3(-3+4\lambda)\right] -2\left[9(4\lambda-3)+6(-\lambda-1)\right]$$

$$= (-7-\lambda)(\lambda^2+2\lambda+1-12\lambda+9) -2(36\lambda-27-6\lambda-6)$$

$$= (-7-\lambda)(\lambda^2-10\lambda+10) -2(30\lambda-33)$$

$$= -7\lambda^2+70\lambda-70-\lambda^3+10\lambda^2-10\lambda-60\lambda+66.$$

$$= -\lambda^3+3\lambda^2-4.$$

$$= -(\lambda+1)(\lambda-2)^2.$$

Eigenvalue $\lambda_1=-1$ multiplicity 1.

eigenvector. $\begin{bmatrix} -6 & 9 & -6 \\ -8 & 12 & -7 \\ -2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 0 \\ -6 & 9 & -6 \\ -8 & 12 & -7 \end{bmatrix} \begin{matrix} 3①+② \\ 4①+③ \end{matrix}$

$\rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$2v_1-3v_2=0 \quad v_3=0. \qquad \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

First fundamental sol. $\vec{x}_1(t) = e^{\lambda t}\vec{v}_1 = e^{-t}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

Eigenvector $\lambda_2=2$ multiplicity $m=2$

Solve $(A-\lambda_2 I)^2\vec{v}=0$ to obtain 2 linearly independent solutions.

$$A-\lambda_2 I = \begin{bmatrix} -9 & 9 & -6 \\ -8 & 9 & -7 \\ -2 & 3 & -3 \end{bmatrix}$$

$$(A-\lambda_2 I)^2 = \begin{bmatrix} 21 & -18 & 9 \\ 14 & -12 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 7 & -6 & 3 \\ 7 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & -6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$7v_1-6v_2+3v_3=0$

$\Rightarrow 3v_3 = -7v_1+6v_2.$

$\Rightarrow v_3 = -\frac{7}{3}v_1+2v_2.$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ -\frac{7}{3}v_1+2v_2 \end{bmatrix} = \frac{1}{3}v_1\begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$\vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$\vec{x}_2 = e^{2t}\left[\vec{v}_2 + t(A-\lambda_2 I)\vec{v}_2\right]$$

$$= e^{2t}\left[\begin{pmatrix} 3 \\ 0 \\ -7 \end{pmatrix} + t\begin{pmatrix} -9 & 9 & -6 \\ -8 & 9 & -7 \\ -2 & 3 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -7 \end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix} 3+15t \\ 25t \\ -7+15t \end{pmatrix}.$$

$$\vec{x}_3(t) = e^{2t}\left[\vec{v}_3 + t(A-\lambda I)\vec{v}_3\right]$$

$$= e^{2t}\left[\begin{pmatrix}0\\1\\2\end{pmatrix} + t\begin{pmatrix}-9 & 9 & -6\\-8 & 9 & -7\\-2 & 3 & -3\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix}-3t\\1-5t\\2-3t\end{pmatrix}$$

General sol. $\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) + C_3\vec{x}_3(t)$

$$= C_1 e^{-t}\begin{bmatrix}3\\2\\0\end{bmatrix} + C_2 e^{2t}\begin{bmatrix}3+15t\\25t\\-7+15t\end{bmatrix} + C_3 e^{2t}\begin{bmatrix}-3t\\1-5t\\2-3t\end{bmatrix}$$

Question 12 Find the fundamental matrix e^{At} where

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}.$$

Solve the homogeneous system

$$\vec{x}'(t) = A\vec{x}(t).$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (3-\lambda)(-1-\lambda)+8.$$
$$= -3-3\lambda+\lambda+\lambda^2+8$$
$$= \lambda^2-2\lambda+5$$

$$\lambda = \frac{2\pm\sqrt{4-4\cdot 5}}{2} = \frac{2\pm\sqrt{-16}}{2}$$
$$= 1\pm 2i$$

When $\lambda_1 = 1+2i$ eigenvector:

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-i)v_1 - v_2 = 0.$$

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

fundamental sol.

$$\vec{x}_1(t) = e^t\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin 2t\right)$$
$$= e^t\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

$$\vec{x}_2(t) = e^t\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos 2t\right)$$
$$= e^t\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}.$$

Let $X(t) = e^t\begin{bmatrix} \cos 2t & \sin 2t \\ \cos 2t + \sin 2t & \sin 2t - \cos 2t \end{bmatrix}$

$$X(0) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\det X(0) = -1$$

$$X(0)^{-1} = -1\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$e^{At} = X(t)X(0)^{-1}$$
$$= e^t\begin{bmatrix} \cos 2t & \sin 2t \\ \cos 2t+\sin 2t & \sin 2t-\cos 2t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$
$$= e^t\begin{bmatrix} \cos 2t+\sin 2t & -\sin 2t \\ 2\sin 2t & \cos 2t-\sin 2t \end{bmatrix}$$