Math 2552 Practice Problems for final

Question 1. Please answer the following short questions and show all your work for full credit.

(1) Find the longest interval in which the solution of the initial value problem

$$\begin{aligned}(t-0.5)\frac{dx}{dt} &= tx + t^2 y\\ (\cos t)\frac{dy}{dt} &= (\sin t)x + \frac{1}{t-2}y\\ x(1) = 1,&\quad y(1) = 2\end{aligned}$$

is certain to exist.

(2) Is the function $f(t) = e^{t^3}$ of exponential order? Why?

(3) For the non-homogeneous equation $y'' + 2y' + y = t^3e^{-t}$, determine a suitable form for the particular solution if the method of underdetermined coefficients is to be used. Do not solve for the coefficients.

(4) Transform the following 4th order linear equation into a system of first order equations

$$y^{(4)} - ty''' + y'' = \sin t.$$

Question 2. Consider the differential equation

$$(1 + xy^3) + cx^2y^2\frac{dy}{dx} = 0.$$

(1) In order for the equation above to be exact, what should c be?

(2) Solve the equation with the c found in (1).

Question 3. Find the general solution of

$$x\frac{dy}{dx} - y = 2x^3, \ x > 0.$$

Question 4. The tank initially contains 100 gal of fresh water. Water containing a salt concentration of $\sin t$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the rate of 1 gal/min. Find the amount of salt in the tank at any time.

Question 5.

(1) Show that $\mathbf{x}_1 = \begin{bmatrix} t \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} t^{\frac{1}{4}} \\ \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix}$ form a fundamental set of solutions for the homogeneous system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix} \mathbf{x},\ t > 0.$$

You need to check that $\mathbf{x_1}$ and $\mathbf{x}_2$ are solutions, and they are linearly independent.

(2) Find the general solution of the following non-homogeneous system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{5}{4t} \end{bmatrix}.$$

Question 6. Consider the system of equations

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & a \end{bmatrix} \mathbf{x}.$$

(1) If $a = 0$, what is the type and stability of the critical point $(0, 0)$? Then find the solution to the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

(2) In order for $(0, 0)$ to be a spiral sink, what interval should a be in? Show your work. If it can never be a spiral sink, explain why.

Question 7. Compute the Laplace transform of the following functions:

(1) $f(t) = te^{-t}u_1(t)$.

(2) $f(t) = \begin{cases} 2 & 0 \leq t < 4; \\ t & t \geq 4. \end{cases}$

Question 8. Compute the Laplace transform of the following functions:

(1) $F(s) = \frac{2-s}{s^2+2s+2}$.

(2) $F(s) = \frac{e^{-3s}}{s(s^2+1)}$.

Question 9. Use the Laplace transform to solve the following equation:

$$y^{(4)} - y = 0$$

with the initial value $y(0) = 1, y'(0) = 0, y''(0) = 1, y'''(0) = 0$.

Question 10. Consider the initial value problem:

$$y'' + 4y = g(t), \quad y(0) = 0,\ y'(0) = 0$$

where

$$g(t) = \begin{cases} e^{2t} & \text{if } t \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

1. Express $g(t)$ in terms of heaviside functions.

2. Compute $\mathcal{L}\{g\}$.

3. Use the Laplace transform to solve the initial value problem.

Question 11. Find the general solution of the following system of equations

$$\mathbf{x}' = \begin{bmatrix} -7 & 9 & -6 \\ -8 & 11 & -7 \\ -2 & 3 & -1 \end{bmatrix} \mathbf{x}.$$

Notice that $\lambda^3 - 3\lambda^2 + 4 = (\lambda + 1)(\lambda - 2)^2$. You need to show your work about the computation of the determinant.

Question 12. Find the fundamental matrix e^{At} where

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}.$$

Elementary Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$
$t^n,\ n =$ positive integer	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin at$	$\frac{a}{s^2+a^2},\ s > 0$
$\cos at$	$\frac{s}{s^2+a^2},\ s > 0$
$u_c(t)$	$\frac{e^{-cs}}{s},\ s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$\delta(t-c)$	e^{-cs}
$t^n f(t)$	$(-1)^n F^{(n)}(s)$