Question 1:

First, solve the homogeneous equation $y'' - 4y' + 4y = 0$.

characteristic eq: $\lambda^2 - 4\lambda + 4 = 0 \quad (\lambda-2)^2 = 0 \quad \lambda = 2$.

Two fundamental sol. $y_1(t) = e^{2t} \quad y_2(t) = te^{2t}$.

Second, use the method of underdetermined coefficients to find a particular sol.

As 2 appears twice in the roots of the characteristic equation, we let

$$Y(t) = At^2 e^{2t} + B.$$

$$Y'(t) = 2Ate^{2t} + 2At^2 e^{2t}$$

$$Y''(t) = 2Ae^{2t} + 4Ate^{2t} + 4Ate^{2t} + 4At^2 e^{2t}$$

$$= 2Ae^{2t} + 8Ate^{2t} + 4At^2 e^{2t}.$$

$$Y'' - 4Y' + 4Y = 2Ae^{2t} + \cancel{8Ate^{2t}} + \cancel{4At^2 e^{2t}}$$

$$- \cancel{8Ate^{2t}} - \cancel{8At^2 e^{2t}} + \cancel{4At^2 e^{2t}} + 4B$$

$$= 2Ae^{2t} + 4B \qquad = e^{2t} + 1.$$

Then $2A = 1 \Rightarrow A = \frac{1}{2}$.

$4B = 1 \Rightarrow B = \frac{1}{4}$

General sol. $y(t) = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{2}t^2 e^{2t} + \frac{1}{4}$.

Question 2

(1) Show that $\mathbf{x}_1 = \begin{bmatrix} t \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} t^{\frac{1}{4}} \\ \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix}$ form a fundamental set of solutions for the homogeneous system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\mathbf{x}, \quad t>0.$$

You need to check that $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions, and they are linearly independent.

$$\frac{d\vec{x}_1}{dt} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\vec{x}_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{d\vec{x}_2}{dt} = \begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} \\ -\frac{3}{16}t^{-\frac{7}{4}} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\begin{bmatrix} t^{1/4} \\ \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} \\ -\frac{3}{16}t^{-\frac{7}{4}} \end{bmatrix}$$

$$W[\vec{x}_1\ \vec{x}_2] = \begin{vmatrix} t & t^{\frac{1}{4}} \\ 1 & \frac{1}{4}t^{-\frac{3}{4}} \end{vmatrix} = \frac{1}{4}t^{\frac{1}{4}} - t^{\frac{1}{4}} = -\frac{3}{4}t^{\frac{1}{4}} \neq 0.$$

Yes, $\vec{x}_1$, $\vec{x}_2$ are solutions and they are linearly independent.

(2) Find the general solution of the following non-homogeneous system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4t^2} & \frac{1}{4t} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{5}{4t} \end{bmatrix}.$$

Let $X(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} t & t^{1/4} \\ 1 & \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix} \qquad \vec{g}(t) = \begin{bmatrix} 0 \\ \frac{5}{4t} \end{bmatrix}$

A particular sol. is $\vec{x}_p(t) = X(t)\vec{u}(t)$ where $\vec{u}' = X(t)^{-1}\vec{g}(t)$.

$$\det X(t) = -\frac{3}{4}t^{\frac{1}{4}} \qquad X(t)^{-1} = \frac{1}{-\frac{3}{4}t^{\frac{1}{4}}}\begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} & -t^{\frac{1}{4}} \\ -1 & t \end{bmatrix}$$

$$= -\frac{4}{3}t^{-\frac{1}{4}}\begin{bmatrix} \frac{1}{4}t^{-\frac{3}{4}} & -t^{\frac{1}{4}} \\ -1 & t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}t^{-1} & \frac{4}{3} \\ \frac{4}{3}t^{-\frac{1}{4}} & -\frac{4}{3}t^{\frac{3}{4}} \end{bmatrix}$$

$$\vec{u}'(t) = \begin{bmatrix} -\frac{1}{3t} & \frac{4}{3} \\ \frac{4}{3}t^{-\frac{1}{4}} & -\frac{4}{3}t^{\frac{3}{4}} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{5}{4t} \end{bmatrix} = \begin{bmatrix} \frac{5}{3t} \\ -\frac{5}{3}t^{-\frac{1}{4}} \end{bmatrix} = \begin{bmatrix} \frac{5}{3}t^{-1} \\ -\frac{5}{3}t^{-\frac{1}{4}} \end{bmatrix}$$

$$\vec{u}(t) = \frac{5}{3}\begin{bmatrix} \ln t \\ -\frac{4}{3}t^{\frac{3}{4}} \end{bmatrix} \qquad \vec{x}_p = \begin{bmatrix} t & t^{1/4} \\ 1 & \frac{1}{4}t^{-\frac{3}{4}} \end{bmatrix}\begin{bmatrix} \frac{5}{3}\ln t \\ -\frac{20}{9}t^{\frac{3}{4}} \end{bmatrix} = \begin{bmatrix} \cancel{\frac{5}{3}t\ln t + \frac{5}{12}t} \\ \cancel{\frac{5}{3}\ln t + \frac{5}{\ldots}} \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} \frac{5}{3} t \ln t - \frac{20}{9} t \\ \frac{5}{3} \ln t - \frac{5}{9} \end{bmatrix}$$

General solution of the non-homogeneous system.

$$\vec{x}(t) = C_1 \begin{bmatrix} t \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} t^{\frac{1}{4}} \\ \frac{1}{4} t^{-\frac{3}{4}} \end{bmatrix} + \begin{bmatrix} \frac{5}{3} t \ln t - \frac{20}{9} t \\ \frac{5}{3} \ln t - \frac{5}{9} \end{bmatrix}$$

Question 3

$$x' = \underbrace{\begin{bmatrix} -7 & 9 & -6 \\ -8 & 11 & -7 \\ -2 & 3 & -1 \end{bmatrix}}_{A} x.$$

$$A-\lambda I = \begin{bmatrix} -7-\lambda & 9 & -6 \\ -8 & 11-\lambda & -7 \\ -2 & 3 & -1-\lambda \end{bmatrix} -4③+②.$$

Elementary row operations $\rightarrow \begin{bmatrix} -7-\lambda & 9 & -6 \\ 0 & -1-\lambda & -3+4\lambda \\ -2 & 3 & -1-\lambda \end{bmatrix}$

$$\det A-\lambda I = (-7-\lambda)\begin{vmatrix} -1-\lambda & -3+4\lambda \\ 3 & -1-\lambda \end{vmatrix} -2 \begin{vmatrix} 9 & -6 \\ -1-\lambda & -3+4\lambda \end{vmatrix}$$

$= (-7-\lambda)\left[(\lambda+1)^2-3(-3+4\lambda)\right] -2\left[9(4\lambda-3)+6(-\lambda-1)\right]$

$= (-7-\lambda)(\lambda^2+2\lambda+1-12\lambda+9) -2(36\lambda-27-6\lambda-6)$

$= (-7-\lambda)(\lambda^2-10\lambda+10)-2(30\lambda-33)$

$= -7\lambda^2+70\lambda-70-\lambda^3+10\lambda^2-10\lambda-60\lambda+66$

$= -\lambda^3+3\lambda^2-4.$

$= -(\lambda+1)(\lambda-2)^2.$

Eigenvalue $\lambda_1=-1$ multiplicity 1.

eigenvector. $\begin{bmatrix} -6 & 9 & -6 \\ -8 & 12 & -7 \\ -2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 0 \\ -6 & 9 & -6 \\ -8 & 12 & -7 \end{bmatrix} \begin{matrix} 3①+② \\ 4①+③ \end{matrix}$

$\rightarrow \begin{bmatrix} 2 & -3 & 0. \\ 0 & 0 & -6 \\ 0 & 0 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$2v_1-3v_2=0 \quad v_3=0. \qquad \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

First fundamental sol. $\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 = e^{-t}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

Eigenvector $\lambda_2=2$ multiplicity $m=2$.

Solve $(A-\lambda_2 I)^2\vec{V}=0$ to obtain 2 linearly independent solutions

$$A-\lambda_2 I = \begin{bmatrix} -9 & 9 & -6 \\ -8 & 9 & -7 \\ -2 & 3 & -3 \end{bmatrix}$$

$$(A-\lambda_2 I)^2 = \begin{bmatrix} 21 & -18 & 9 \\ 14 & -12 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 7 & -6 & 3 \\ 7 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & -6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$7v_1-6v_2+3v_3=0$

$\Rightarrow 3v_3 = -7v_1+6v_2.$

$\Rightarrow v_3 = -\frac{7}{3}v_1+2v_2.$

$$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ -\frac{7}{3}v_1+2v_2 \end{bmatrix} = \frac{1}{3}v_1\begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix} \qquad \vec{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$\vec{x}_2 = e^{2t}\left[\vec{V}_2 + t(A-\lambda_2 I)\vec{V}_2\right]$

$= e^{2t}\left[\begin{pmatrix} 3 \\ 0 \\ -7 \end{pmatrix} + t\begin{pmatrix} -9 & 9 & -6 \\ -8 & 9 & -7 \\ -2 & 3 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -7 \end{pmatrix}\right]$

$= e^{2t}\begin{pmatrix} 3+15t \\ 25t \\ -7+15t \end{pmatrix}.$

$$\vec{x}_3(t) = e^{2t}\left[\vec{v}_3 + t(A-\lambda I)\vec{v}_3\right]$$

$$= e^{2t}\left[\begin{pmatrix}0\\1\\2\end{pmatrix} + t\begin{pmatrix}-9 & 9 & -6\\-8 & 9 & -7\\-2 & 3 & -3\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix}-3t\\1-5t\\2-3t\end{pmatrix}$$

General sol. $\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) + C_3\vec{x}_3(t)$

$$= C_1 e^{-t}\begin{bmatrix}3\\2\\0\end{bmatrix} + C_2 e^{2t}\begin{bmatrix}3+15t\\25t\\-7+15t\end{bmatrix} + C_3 e^{2t}\begin{bmatrix}-3t\\1-5t\\2-3t\end{bmatrix}$$

Question 4

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}.$$

Solve the homogeneous system

$$\vec{x}'(t) = A\vec{x}(t)$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (3-\lambda)(-1-\lambda) + 8.$$
$$= -3 - 3\lambda + \lambda + \lambda^2 + 8$$
$$= \lambda^2 - 2\lambda + 5$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$
$$= 1 \pm 2i$$

When $\lambda_1 = 1+2i$ eigenvector.

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-i)v_1 - v_2 = 0.$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Fundamental sol.

$$\vec{x}_1(t) = e^t\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin 2t\right)$$
$$= e^t\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

$$\vec{x}_2(t) = e^t\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos 2t\right)$$
$$= e^t\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}.$$

Let $X(t) = e^t\begin{bmatrix} \cos 2t & \sin 2t \\ \cos 2t + \sin 2t & \sin 2t - \cos 2t \end{bmatrix}$

$$X(0) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\det X(0) = -1$$

$$X(0)^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$e^{At} = X(t)X(0)^{-1}$$

$$= e^t\begin{bmatrix} \cos 2t & \sin 2t \\ \cos 2t + \sin 2t & \sin 2t - \cos 2t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= e^t\begin{bmatrix} \cos 2t + \sin 2t & -\sin 2t \\ 2\sin 2t & \cos 2t - \sin 2t \end{bmatrix}$$

$$\cancel{= e^t\begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}}$$