Math 2552 Practice Problems for Midterm 1

Question 1. (a) Solve the initial value problem

$$\frac{dy}{dx} = x^2(y - 4xy), \quad y(0) = 10.$$

(b) Find an interval in which the initial value problem

$$(\sin t)y' + ty = 2\ln(t-1), \quad y(2) = 1$$

has a unique solution. You must clear about your reasoning.

(a). Separable equation

$$\frac{dy}{dx} = yx^2(1-4x).$$

$$\int \frac{dy}{y} = \int x^2(1-4x)\,dx$$

$$\ln|y| = \frac{1}{3}x^3 - x^4 + C$$

Plug in initial value. $x=0$, $y=10$.

$$\ln 10 = C.$$

Solution

$$\ln|y| = \frac{1}{3}x^3 - x^4 + \ln 10.$$

Remark: No need to write an explicit solution if not specified.

(b). This is a first order linear equation.

The standard form is

$$y' + \frac{t}{\sin t} y = \frac{2\ln(t-1)}{\sin t}.$$

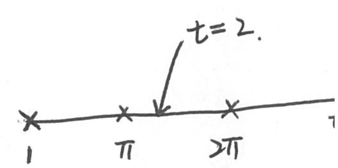
Let $p(t) = \frac{t}{\sin t}$ $\quad g(t) = \frac{2\ln(t-1)}{\sin t}$.

$p(t)$ and $g(t)$ are not defined at

$t<1$ $\quad t = 0, \pi, 2\pi, 3\pi, 4\pi, \cdots$

and $\quad t = -\pi, -2\pi, -3\pi, \cdots$

$t>1$:



The interval is $(\pi, 2\pi)$.

Question 2. Solve the initial value problem

$$(y^2 + \cos x) + (2xy + 2y)\frac{dy}{dx} = 0, \quad y(0) = -2,$$

and express your solution as an **explicit function**. In what interval does the solution exist?

The differential equation is equivalent to.

$$\underbrace{(y^2 + \cos x)}_{M(x,y)} dx + \underbrace{(2xy + 2y)}_{N(x,y)} dy = 0.$$

$\frac{\partial M}{\partial y} = 2y.$ $\quad \frac{\partial N}{\partial x} = 2y$ $\quad$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exac

We need to find ϕ such that

$$\frac{\partial \phi}{\partial x} = M \qquad \frac{\partial \phi}{\partial y} = N.$$

First $\phi = \int N\, dy + h(x) = \int (2xy + 2y)\, dy + h(x)$

$$= xy^2 + y^2 + h(x).$$

Then plug ϕ into $\frac{\partial \phi}{\partial x} = M.$

$$\frac{\partial \phi}{\partial x} = y^2 + h'(x) = y^2 + \cos x \Rightarrow \int h'(x) = \int \cos x$$

$$\Rightarrow h(x) = \sin x + C.$$

Then $\phi(x,y) = xy^2 + y^2 + \sin x + C.$

The solution is $xy^2 + y^2 + \sin x + C = 0.$

Initial value. $x = 0.$ $y = -2$ $\quad 4 + C = 0 \quad C = -4.$

$$xy^2 + y^2 + \sin x - 4 = 0.$$

Explicit form. $y^2 = \dfrac{4 - \sin x}{x + 1}.$

Since the initial value is $x=0$ $y=-2<0$.

$$\boxed{y=-\sqrt{\frac{4-\sin x}{x+1}}}$$

In what interval does the solution exist?

we need. $\frac{4-\sin x}{x+1} \geq 0$. Since $4-\sin x$ is always positive.

we need $x+1>0$

$\therefore x>-1$

So solution exists. in $(-1,+\infty)$.

Question 3. In a container with 20 gallon capacity, there is initially 10 gallons of fresh water. A brine solution containing 0.25 lb/gal of salt flows into the container at a rate of 4 gal/min. The solution is kept thoroughly mixed, and the mixture flows out at a rate of 2 gal/min. How much salt is in the container at the moment it overflows?

Let $y(t)$ be the amount of salt (in lbs) in the container.

at time t (in minutes).

Interal condition $y(0) = 0$.

$y'(t)$ = rate in − rate out.

0.25×4. $\quad 2 \times \dfrac{y}{\text{Volume of liquid at } t}$.

Volume of Liquid at $t = 10 + 4t - 2t \Rightarrow$ Overflows at $t=5$

$$y'(t) = 1 - \frac{2y}{10+2t}$$ First order linear equation.

$$y'(t) + \frac{1}{5+t} y(t) = 1.$$

Integrating factor $\mu(t) = e^{\int \frac{1}{5+t} dt} = e^{\ln(5+t)} = 5+t$

$$\frac{d}{dt}\left(\mu(t) y(t)\right) = \mu(t) \cdot 1 = 5+t.$$

$$\mu(t) y(t) = \int 5+t \, dt = 5t + \tfrac{1}{2}t^2 + C.$$

$$y(t) = \frac{5t + \frac{1}{2}t^2 + C}{5+t}.$$

Interal value $t=0$ $y=0$ $\quad \frac{C}{5} = 0 \quad C = 0.$

Solution $$y(t) = \frac{5t + \frac{1}{2}t^2}{5+t}$$

When it overflows at $t=5$, $$y(5) = \frac{5 \cdot 5 + \frac{1}{2} \cdot 25}{5+5} = \frac{25 + \frac{25}{2}}{10} = \frac{75}{2} \cdot \frac{1}{10} = \frac{15}{4}$$

Question 4. Let $y(t)$ denote the population of a certain species of fish (in thousands) in the sea at time t (in year). In the absence of other factors, assume that y satisfies the logistic equation:

$$\frac{dy}{dt} = y(4-y).$$

(a) In addition to the logistic equation, assume that k (thousands of) fish are consumed by human beings each year continuously. Write down a new differential equation describing the fish population in the sea.

(b) If $k = 3$, find all equilibrium solutions for the new model in (a), and use the **phase line method** to determine their stability.

(c) Assume $y(0) = 2$. For the model in (a), if we do not want the fish to extinct, how big can k be at most? Explain your reasoning.

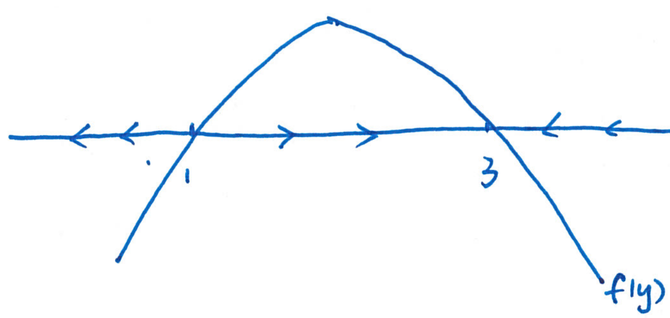
(a). $\frac{dy}{dt} = y(4-y) - k.$

(b) When $k=3$ $\quad \frac{dy}{dt} = \underbrace{4y - y^2 - 3}_{f(y)} = -(y^2-4y+3) = -(y-1)(y-3)$

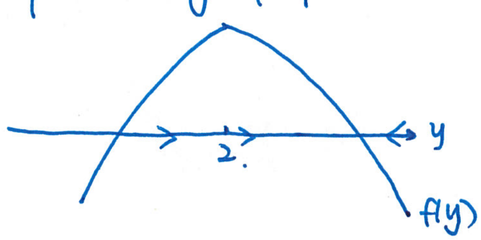
Equilibrium solutions. $\quad y = 1 \quad$ unstable

$y = 3 \quad$ asymptotically stable.

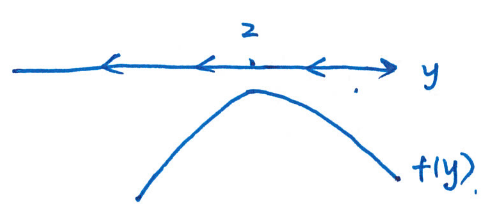
Phase line



(c). f(y) is a parabola facing down and symmetric about $y=2$.
Some possibility of phase line.



Fish does not extinct



Fish will extinct.

Fish does not extinct if $f(2) \geq 0$.

$$4 \cdot 2 - 2^2 - k \geq 0.$$

$$4 - k \geq 0$$

$$\boxed{k \leq 4}$$

Question 5. Consider the system of equations

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}}_{A} \mathbf{x}.$$

(a) Find the general solution of the system.

(b) Sketch a phase portrait. What is the type of the critical point $(0,0)$?

(a) Find eigenvalue & eigenvector of A

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} \qquad \det(A-\lambda I) = (1-\lambda)(-4-\lambda) - (-6)$$
$$= -4 - \lambda + 4\lambda + \lambda^2 + 6 = \lambda^2 + 3\lambda + 2.$$
$$= (\lambda+2)(\lambda+1) = 0.$$

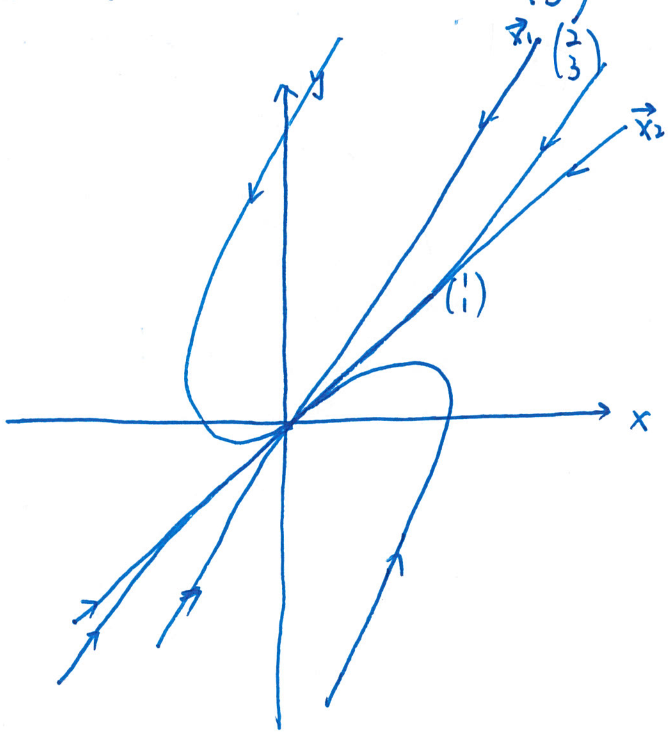
$$\lambda_1 = -2 \qquad \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \vec{x}_1(t) = e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\lambda_2 = -1 \qquad \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$W[\vec{x}_1, \vec{x}_2](t) = \det\begin{pmatrix} 2e^{-2t} & e^{-t} \\ 3e^{-2t} & e^{-t} \end{pmatrix} = 2e^{-3t} - 3e^{-3t} = -e^{-3t} \neq 0.$$

General sol. $\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(b)



(0,0) is a stable node.