Math 2552 Practice Problems for Midterm 1

Question 1. (a) Solve the initial value problem

$$\frac{dy}{dx} = x^2(y - 4xy), \quad y(0) = 10.$$

(b) Find an interval in which the initial value problem

$$(\sin t)y' + ty = 2\ln(t - 1), \quad y(2) = 1$$

has a unique solution. You must clear about your reasoning.

Question 2. Solve the initial value problem

$$(y^2 + \cos x) + (2xy + 2y)\frac{dy}{dx} = 0, \quad y(0) = -2,$$

and express your solution as an **explicit function**. In what interval does the solution exist?

Question 3. In a container with 20 gallon capacity, there is initially 10 gallons of fresh water. A brine solution containing 0.25 lb/gal of salt flows into the container at a rate of 4 gal/min. The solution is kept thoroughly mixed, and the mixture flows out at a rate of 2 gal/min. How much salt is in the container at the moment it overflows?

Question 4. Let $y(t)$ denote the population of a certain species of fish (in thousands) in the sea at time t (in year). In the absence of other factors, assume that y satisfies the logistic equation:

$$\frac{dy}{dt} = y(4 - y).$$

(a) In addition to the logistic equation, assume that k (thousands of) fish are consumed by human beings each year continuously. Write down a new differential equation describing the fish population in the sea.

(b) If $k = 3$, find all equilibrium solutions for the new model in (a), and use the **phase line method** to determine their stability.

(c) Assume $y(0) = 2$. For the model in (a), if we do not want the fish to extinct, how big can k be at most? Explain your reasoning.

Question 5. Consider the system of equations

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}.$$

(a) Find the general solution of the system.

(b) Sketch a phase portrait. What is the type of the critical point $(0,0)$?