Chapter 6.

$$x_1'(t) = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t).$$

$$x_2'(t) = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t).$$

$$\vdots$$

$$x_n'(t) = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t).$$

Denote $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}_{n\times 1}$ $\quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & & & \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}$ $\quad \vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$.

$$\frac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t) + \vec{g}(t)$$

Some terminology

- This system is homogeneous if $\vec{g}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.
- " " nonhomogeneous if $\vec{g}(t) \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.
- $\vec{g}(t)$ is often referred to as the input, or forcing function.

Some rules for matrix differentiation.

$$\frac{d}{dt}(C\,P(t)) = C\frac{dP}{dt} \quad \text{where } C \text{ is a constant matrix}$$

$$\frac{d}{dt}\left(P(t) + Q(t)\right) = \frac{dP(t)}{dt} + \frac{dQ(t)}{dt}.$$

$$\frac{d}{dt}\left(P(t)\,Q(t)\right) = P(t)\frac{dQ(t)}{dt} + \frac{dP(t)}{dt}Q(t).$$

Example.

$$x_1' = x_1 + x_2 - x_3$$
$$x_2' = x_1 - x_2$$
$$x_3' = x_2 + x_3$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Linear n-th order. equation.

$$y^{(n)} + p_1(t)\, y^{(n-1)} + P_2\, y^{(n-2)} + \cdots + P_{n-1}\, y' + P_n(t)\, y = g(t)$$

we can transform it into a system of linear equations:

$$x_1 = y. \qquad x_1' = x_2$$
$$x_2 = y' \qquad x_2' = x_3$$
$$x_3 = y'' \qquad x_3' = x_4$$
$$\vdots \qquad \vdots$$
$$x_{n-1} = y^{(n-2)} \qquad x_{n-1}' = x_n.$$
$$x_n = y^{(n-1)} \qquad x_n' = y^{(n)}.$$

Especially. $x_n' = -P_n(t)\, x_1 - P_{n-1} x_2 - \cdots - P_2 x_{n-1} - P_1 x_n + g(t)$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -P_n & -P_{n-1} & -P_{n-2} & \cdots & -P_1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

Example. $t y''' + (\sin t) y'' + 8y = \cos t.$

$$y''' + \frac{\sin t}{t} y'' + \frac{8}{t} y = \frac{\cos t}{t}.$$

Let $x_1 = y \qquad x_2 = y' \qquad x_3 = y''$

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = y''' = -\frac{\sin t}{t} y'' - \frac{8}{t} y + \frac{\cos t}{t}$$
$$= -\frac{8}{t} x_1 - \frac{\sin t}{t} x_3 + \frac{\cos t}{t}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{8}{t} & 0 & -\frac{\sin t}{t} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{\cos t}{t} \end{bmatrix}$$

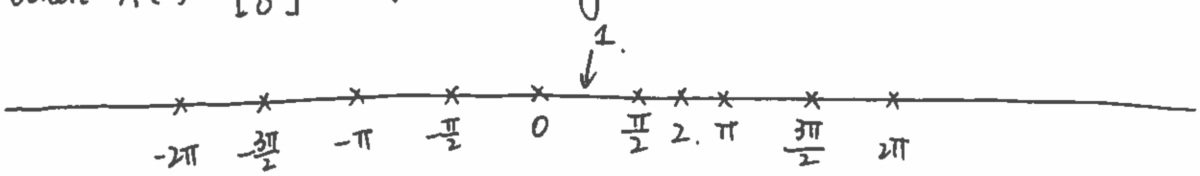
Section 6.2. Basic theory of first order linear systems.

Existence & Uniqueness. $\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{g}(t). \qquad \vec{x}(t_0) = \vec{x}_0$

Theorem 6.2.1 If $P(t)$ and $\vec{g}(t)$ are continuous on an open interval I containing t_0, then the solution exists and is unique throughout the interval I.

EX: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{1}{t} & 0 & \tan t \\ 3 & 1 & \ln|t| \\ 2 & 5 & \cot t \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ 5 \\ \frac{1}{t-2} \end{bmatrix}$

When $\vec{x}(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, find the largest interval where sol. exists.



Answer: $(0, \frac{\pi}{2})$.

Homogeneous eq: $\vec{x}' = P(t)\vec{x}$

Superposition principle: If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are solutions of the homogeneous system, then the linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k$

Proof: Given $\vec{x}_1' = P(t)\vec{x}_1 \quad \vec{x}_2' = P(t)x_2 \quad \cdots \quad \vec{x}_k' = P(t)x_k$.

$$\text{Then } (c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k)' = c_1\vec{x}_1' + c_2\vec{x}_2' + \cdots + c_k\vec{x}_k'$$
$$= c_1P(t)\vec{x}_1 + c_2P(t)\vec{x}_2 + \cdots + c_kP(t)\vec{x}_k$$
$$= P(t)\left[c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k\right]$$

vector function.

Def [linearly independent] $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent if.

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t) = \vec{0} \text{ for all } t \quad \text{implies} \quad c_1 = c_2 = \cdots = c_n = 0.$$

If there exist constants $c_1, c_2, \cdots c_n$ not all zero, such that.

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t) = \vec{0} \text{ for all } t.$$

then $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n(t)$ are linearly dependent.

How to determine. a set of vector functions is linearly independent.

Step 1. Form the system $c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t) = \vec{0}$

Step 2. Solve for $c_1, c_2, \cdots, c_n$.

If there is a unique sol. $c_1 = c_2 = \cdots = c_n = 0$, linearly independent

If there are non-zero sol. linearly dependent.

Ex. $\vec{x}_1(t) = \begin{pmatrix} e^{-2t} \\ 0 \\ -e^{-2t} \end{pmatrix}$ $\quad \vec{x}_2 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$

Step1. $c_1 \begin{pmatrix} e^{-2t} \\ 0 \\ -e^{-2t} \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for all t.

Step2. Let $t=0$ $\quad c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad c_1 = 0$, $c_2 = 0$

linearly independent.

Ex. $\vec{x}_1(t) = \begin{pmatrix} 1+t \\ t \\ 1-t \end{pmatrix}$ $\quad \vec{x}_2(t) = \begin{pmatrix} 3 \\ t+2 \\ t \end{pmatrix}$ $\quad \vec{x}_3(t) = \begin{pmatrix} 1-2t \\ 2-t \\ 3t-2 \end{pmatrix}$

Step1. $c_1 \begin{pmatrix} 1+t \\ t \\ 1-t \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ t+2 \\ t \end{pmatrix} + c_3 \begin{pmatrix} 1-2t \\ 2-t \\ 3t-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$c_1(1+t) + 3c_2 + c_3(1-2t) = 0.$$
$$c_1 t + c_2(t+2) + c_3(2-t) = 0.$$
$$c_1(1-t) + c_2 t + c_3(3t-2) = 0.$$

$c_1 \quad -2c_3 = 0.$ $\qquad c_1 = 2c_3$

$c_1 + 3c_2 + c_3 = 0.$ $\qquad 2c_3 + 3c_2 + c_3 = 0.$ $\qquad 3c_2 + 3c_3 = 0.$

$c_1 + c_2 - c_3 = 0.$ $\qquad 2c_3 + c_2 - c_3 = 0.$ $\qquad c_2 + c_3 = 0$ $\qquad c_2 = -c_3$

$2c_2 + 2c_3 = 0.$ $\qquad 2c_2 + 2c_3 = 0.$ $\qquad c_2 + c_3 = 0$

$-c_1 + c_2 + 3c_3 = 0.$ $\qquad -2c_3 + c_2 + 3c_3 = 0.$ $\qquad c_2 + c_3 = 0.$

$\cancel{c_1 \quad -2c_3 = 0}$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2c_3 \\ -c_3 \\ c_3 \end{pmatrix}$$

Linearly dependent.

Def [Wronskian]. Let $\vec{x}_1, \dots, \vec{x}_n$ be solutions of $\vec{x}\,' = \underbrace{P(t)}_{n\times n}\underbrace{\vec{x}}_{n\times 1}$.

$$\vec{x}_1 = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \quad \dots, \quad \vec{x}_n = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

$$X(t) = [\vec{x}_1 \ \cdots \ \vec{x}_n] = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{bmatrix}$$

The Wronskian $W = W[\vec{x}_1 \cdots \vec{x}_n] = \det X(t)$.

Theorem 6.2.5. Let $\vec{x}_1, \cdots \vec{x}_n$ be solutions of $\vec{x}\,' = P(t)\vec{x}$

(i) if $\vec{x}_1, \dots, \vec{x}_n$ are linearly independent, then $W[\vec{x}_1, \dots, \vec{x}_n](t) \neq 0$ for every t.

(ii) if $\vec{x}_1, \dots, \vec{x}_n$ are linearly dependent, then $W[\vec{x}_1, \dots, \vec{x}_n](t) = 0$ for every t.

Fundamental set of solutions.

Thm. 6.2.6 Let $\vec{x}_1, \dots, \vec{x}_n$ be solutions of $\frac{d\vec{x}}{dt} = P(t)\vec{x}$ such that $W[\vec{x}_1, \dots, \vec{x}_n](t) \neq 0$. Then the general solution of $\frac{d\vec{x}}{dt} = P(t)\vec{x}$

is $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$.

$\vec{x}_n, \dots, \vec{x}_n$ is said to be a fundamental set of solutions.

Ex. $\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & -1 & 2 \\ 2 & -3 & 2 \\ 3 & -3 & 1 \end{pmatrix} \vec{x}$ $\quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Show that $\vec{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad \vec{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad \vec{x}_3(t) = e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

form a fundamental set of solution.

$\rightarrow$ Check the Wronskian.

$$W[\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n](t) = \det\left[e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \ e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \ e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]$$

$$= e^{-2t}e^{-t}e^{t} \det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = e^{-2t}\det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

expand along the first column.

$$= e^{-2t}\left(1\cdot\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right) = e^{-2t}(1-0) = e^{-2t} \neq 0$$

Yes, $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ are linearly independent.

Linear nth order equations.

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

$y(t_0)=y_0$
$y'(t_0)=y_1$
$\vdots$
$y^{(n-1)}(t_0)=y_{n-1}$

It is equivalent to system of linear equation.

if $x_1=y$ $\quad x_2=y'$ $\quad x_3=y''$ $\quad \cdots\cdots$ $\quad x_n=y^{(n-1)}$

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ \vdots & & & \ddots & & \vdots \\ & & & & & 1 \\ -p_n(t) & -p_{n-1}(t) & \cdots & & \cdots & -p_1(t) \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix} \qquad \vec{x}(t_0) = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

Existence and uniqueness.

Corollary. If the functions $p_1(t), p_2(t), \cdots, p_n(t)$ and $g(t)$ are continuous on an interval (α,β) containing t_0, then solution to the nth order linear equation exists and is unique on (α,β).

Consider the homogeneous equation.

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots\cdots + p_{n-1}(t)y' + p_n(t)y = 0.$$

Let $y_1(t), y_2(t), \cdots, y_n(t)$ be (scalar) solutions.

<u>Wronskian</u> $\quad W[y_1, \cdots, y_n](t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$

<u>Linearly independent solutions</u> and fundamental set of solutions. $\quad y_1(t), y_2(t) \cdots y_n(t)$ are linearly independent. if $W[y_1\, y_2 \cdots y_n](t) \neq 0$.

<u>General solution</u> $\quad y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$

Ex. Consider. $x^3y''' + x^2y'' - 2xy' + 2y = 0.$

Show that $y_1(x) = x$ $\quad y_2(x) = x^{-1}$ $\quad y_3(x) = x^2$. form a fundamental set of solution.

$y_1'(x) = 1 \quad y_2'(x) = -x^{-2} \quad y_3'(x) = 2x$

$y_1''(x) = 0 \quad y_2''(x) = 2x^{-3} \quad y_3''(x) = 2.$

$$W[y_1\ y_2\ y_3](t) = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{vmatrix}$$

$$= x \begin{vmatrix} -x^{-2} & 2x \\ 2x^{-3} & 2 \end{vmatrix} - \begin{vmatrix} x^{-1} & x^2 \\ 2x^{-3} & 2 \end{vmatrix}$$

$$= x(-2x^{-2} - 4x^{-2}) - (2x^{-1} - 2x^{-1})$$

$$= -6x^{-1} \neq 0.$$

Section 6.3. Homogeneous linear systems with constant coefficients.

- Focus on $\frac{d\vec{x}}{dt} = A\vec{x}$ where A is a constant matrix.

Let λ be an eigenvalue of A and $\vec{v}$ be the associated eigenvetor.

Then $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution. $\quad A\vec{v} = \lambda\vec{v}$

Why $\frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v} = A(e^{\lambda t}\vec{v}) = A\vec{x}$

Three cases to discuss.

1. A has a complete set of n linearly independent eigenvectors and all eigenvalues are real.
nondefective

2. A has a complete set of n linearly independent eigenvectors and one or more complex conjugate pairs of eigenvalues.
nondefective.

3. A is defective, that is, there are one or more eigenvalues of A for which the geometric multiplicity is less than the algebraic multiplicity.

Case 1 when A is nondefective with real eigenvalues.

Thm 6.3.1 Let $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$ be eigenpairs. Assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.

Then $\{e^{\lambda_1 t}\vec{v}_1, e^{\lambda_2 t}\vec{v}_2, \ldots, e^{\lambda_n t}\vec{v}_n\}$ is a fundamental set of solutions to $\vec{x}' = A\vec{x}$. The general solution of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + C_n e^{\lambda_n t}\vec{v}_n.$$

Proof. $W[e^{\lambda_1 t}\vec{v}_1, e^{\lambda_2 t}\vec{v}_2, \ldots, e^{\lambda_n t}\vec{v}_n] = e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t} \underbrace{\det[\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]}_{\neq 0 \text{ because of linear independence}}$

$\neq 0.$

Ex. Find the general solution of.

$$\begin{aligned} x_1' &= x_1 + 4x_2 + 4x_3 \\ x_2' &= 3x_2 + 2x_3 \\ x_3' &= 2x_2 + 3x_3 \end{aligned}$$

Write it in matrix form

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{pmatrix} 1 & 4 & 4 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix}}_{A.} \vec{x} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Compute eigenvalues and eigenvectors of A.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 4 & 4 \\ 0 & 3-\lambda & 2 \\ 0 & 2 & 3-\lambda \end{pmatrix} \qquad \det(A-\lambda I) = (1-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(3-\lambda)^2 - 4\right]$$

$$= (1-\lambda)(\lambda^2 - 6\lambda + 9 - 4)$$

$$= (1-\lambda)(\lambda^2 - 6\lambda + 5)$$

$$= -(\lambda-1)^2(\lambda-5)$$

$\lambda_1 = 1$

$$\begin{pmatrix} 0 & 4 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_2 + v_3 = 0$
$\Rightarrow v_2 = -v_3$.

$$\vec{v} = \begin{pmatrix} v_1 \\ -v_3 \\ v_3 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 5$

$$\begin{pmatrix} -4 & 4 & 4 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-v_1 + v_2 + v_3 = 0.$
$v_2 = v_3.$
Then $v_1 = 2v_3$

$$\vec{v} = \begin{pmatrix} 2v_3 \\ v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Fundamental sol.

$$e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e^t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad e^{5t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

General Sol.

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Special case: real and distinct eigenvalues.

$A_{n\times n}$ has n distinct eigenvalues. $\lambda_1, \lambda_2, \cdots \lambda_n$ where $\lambda_i \neq \lambda_j \ \forall i \neq j$.

The associated eigenvectors are $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$.

FACT: eigenvectors associated with distinct eigenvalues are linearly independent.

So $\{e^{\lambda_1 t}\vec{v_1}, e^{\lambda_2 t}\vec{v_2}, \ldots, e^{\lambda_n t}\vec{v_n}\}$ form a fundamental sol. set.

Ex. Find the general sol. of

$$\vec{x}' = \underbrace{\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}}_{A} \vec{x}$$

$$A-\lambda I = \begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (-1-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(\lambda^2+3\lambda) = -\lambda(\lambda+1)(\lambda+3)$$

$\lambda_1 = 0$.
$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_1 = v_2 \quad v_3 = v_3$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_1 \\ v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\lambda_2 = -1$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$v_2 = 0$. $\quad v_1 + v_3 = 0 \quad \vec{v} = \begin{pmatrix} v_1 \\ 0 \\ -v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$\lambda_3 = -3$ $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$v_1 - v_3 = 0 \quad v_2 + 2v_3 = 0 \quad v_1 = v_3 \quad v_2 = -2v_3$

$$\vec{v} = \begin{pmatrix} v_3 \\ -2v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

General sol. $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$

Homogeneous equation. $\vec{x}' = A\vec{x}$

If there is a pair of complex eigenvalues. $\mu + \nu i$ $\quad$ $\mu - \nu i$

eigenvectors. $\vec{a} + \vec{b} i$ $\quad$ $\vec{a} - \vec{b} i$

Fundamental sol. $\vec{x}_1(t) = e^{\mu t}(\vec{a}\cos\nu t - \vec{b}\sin\nu t)$

$\vec{x}_2(t) = e^{\mu t}(\vec{a}\sin\nu t + \vec{b}\cos\nu t)$.

Ex. Problem 1.

Find general sol. of

$$x_1' = -2x_1 + 2x_2 + x_3$$
$$x_2' = -2x_1 + 2x_2 + 2x_3$$
$$x_3' = 2x_1 - 3x_2 - 3x_3$$

$$A = \begin{pmatrix} -2 & 2 & 1 \\ -2 & 2 & 2 \\ 2 & -3 & -3 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -2-\lambda & 2 & 1 \\ -2 & 2-\lambda & 2 \\ 2 & -3 & -3-\lambda \end{pmatrix}.$$

Add 2nd row to 3rd

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 2 & 1 \\ -2 & 2-\lambda & 2 \\ 0 & -1-\lambda & -1-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 2 & -1 \\ -2 & 2-\lambda & \lambda \\ 0 & -1-\lambda & 0 \end{vmatrix}$$

Add $-1 \times$ 2nd column to 3rd col.

Expand along last row. $\stackrel{}{=} (-1)^{3+2}(-1-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ -2 & \lambda \end{vmatrix}$

$$= (\lambda+1)\Big(\lambda(-\lambda-2)-2\Big) = (\lambda+1)(-\lambda^2-2\lambda-2) = -(\lambda+1)(\lambda^2+2\lambda+2).$$

$$\lambda_1 = -1 \qquad \lambda_2, \lambda_3 = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm 2i}{2} = -1 \pm i$$

Eigenvectors. $\lambda_1 = -1$ $\begin{pmatrix} -1 & 2 & 1 \\ -2 & 3 & 2 \\ 2 & -3 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & 1 \\ -2 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ -2 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\to \begin{pmatrix} 1 & -2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$V_2 = 0$ $V_1 = V_3$ $\vec{V_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Eigenvectors $\lambda_2 = -1+i$ $\begin{pmatrix} -1-i & 2 & 1 \\ -2 & 3-i & 2 \\ 2 & -3 & -2-i \end{pmatrix} \to \begin{pmatrix} -1-i & 2 & 1 \\ -2 & 3-i & 2 \\ 0 & -i & -i \end{pmatrix}$

$$\to \begin{pmatrix} -2 & 3-i & 2 \\ -1-i & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{3}{2}+\frac{i}{2} & -1 \\ -1-i & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \times (1+i)$$

$\frac{(-3+i)(1+i)}{2} = -2-i$

$$\to \begin{pmatrix} 1 & \frac{-3+i}{2} & -1 \\ 0 & -i & -i \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{-3+i}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \times \frac{3-i}{2}$$

$$\to \begin{pmatrix} 1 & 0 & \frac{1}{2}-\frac{i}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$V_1 + \frac{1-i}{2} V_3 = 0$ $V_1 = -\frac{1-i}{2} V_3 = \frac{-1+i}{2} V_3$

$V_2 = -V_3$ let $V_3 = 2$.

$$\vec{V} = \begin{pmatrix} -1+i \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x}(t) = c_1 e^{-t}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2 e^{-t}\left[\begin{pmatrix}-1\\-2\\2\end{pmatrix}\cos t - \begin{pmatrix}1\\0\\0\end{pmatrix}\sin t\right]$$

$$+ c_3 e^{-t}\left[\begin{pmatrix}-1\\-2\\2\end{pmatrix}\sin t + \begin{pmatrix}1\\0\\0\end{pmatrix}\cos t\right]$$

Section 6.5 Fundamental matrices and the exponential of a matrix

Recall the scalar initial value problem.

$$x' = ax \qquad x(0) = x_0. \qquad \text{where } a \text{ is a constant.}$$

Solution: $x(t) = x_0 e^{at}$.

Consider the initial value problem for an $n\times n$ system.

$$\vec{x}\,' = A\vec{x} \qquad \vec{x}(0) = \vec{x}_0$$

We expect to have solution $\vec{x} = \underset{n\times n}{\Phi(t)}\underset{n\times 1}{\vec{x}_0}$

where $\Phi(t)$ is an $n\times n$ matrix satisfying $\Phi(0) = \underbrace{I_{n\times n}}_{\text{identity matrix}}$

How to find this matrix $\Phi(t)$?

Ex.1 $\quad X' = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{A} \vec{x}.$

Eigenvalues & eigenvectors of A:

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} \qquad \det(A-\lambda I) = (1-\lambda)^2 - 4 = (1-\lambda+2)(1-\lambda-2)$$
$$= (3-\lambda)(-1-\lambda)$$

$\lambda = 3 \qquad \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} -2v_1 + v_2 = 0 \\ v_2 = 2v_1 \end{matrix} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

$\lambda_2 = -1 \qquad \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 2v_1 + v_2 = 0 \\ v_2 = -2v_1 \end{matrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Fundamental solutions. $\quad \vec{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \vec{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

Define the matrix $X(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$

General solution $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = X(t)\,\vec{c}$

where $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Plug in initial value $\vec{x}(0) = \underbrace{X(t_0)}_{2\times2}\,\underbrace{\vec{c}}_{2\times1} = \underbrace{\vec{x}_0}_{2\times1}$ then $\vec{c} = X(t_0)^{-1}\,\vec{x}_0$

Solution of the initial value problem. $\vec{x}(t) = \underbrace{X(t)}_{2\times2}\,\underbrace{X(t_0)^{-1}}_{2\times2}\,\underbrace{\vec{x}_0}_{2\times1}$

Therefore $\Phi(t) = X(t)\,X(t_0)^{-1}$

In Ex1. $X(t_0) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

Compute $X(t_0)^{-1}$ $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{array}\right] \times \left(-\tfrac{1}{4}\right)$

$$\to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

$$\Phi(t) = X(t)\,X(t_0)^{-1} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}$$

How to define $\Phi(t)$ in general?

In scalar case. $\Phi(t) = e^{at} = 1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \cdots$

$$= \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

In vector case, $\Phi(t) = e^{At}$??

Def. [Matrix exponential function]

Let A be an $n \times n$ matrix.

$$e^{At} = I_n + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} A^k \frac{t^k}{k!}$$

Ex2. $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$. Compute e^{At}.

$$e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{pmatrix}.$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k t^k}{k!} & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\lambda_2^k t^k}{k!} & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{\lambda_3^k t^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}$$

Ex3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Compute e^{At}.

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad \cdots \quad A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & \sum_{k=0}^{\infty} \frac{t^k k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} \end{pmatrix}$$

$$\sum_{k=0}^{\infty} \frac{t^k k}{k!} = \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!} = t \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \overset{l=k-1}{=\!=} t \sum_{l=0}^{\infty} \frac{t^l}{l!} = te^t.$$

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So $e^{At} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$.

Why do we define e^{At}?

Consider the initial value problem. $\vec{x}' = A\vec{x}$ $\quad \vec{x}(0) = \vec{x_0}$

Then the solution is $\vec{x}(t) = e^{At}\vec{x_0}$ [Theorem 6.5.2]

Proof: $\frac{d\vec{x}(t)}{dt} = \left(\frac{d}{dt} e^{At}\right)\vec{x_0}$

$$e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

$$\frac{d}{dt} e^{At} = \sum_{k=0}^{\infty} A^k \frac{kt^{k-1}}{k!} = \sum_{k=1}^{\infty} A^k \frac{t^{k-1}}{(k-1)!} = A \sum_{k=1}^{\infty} A^{k-1} \frac{t^{k-1}}{(k-1)!}$$

By letting $l = k-1$

$$= A \sum_{l=0}^{\infty} A^l \frac{t^l}{l!} = Ae^{At}.$$

$$\boxed{\frac{d}{dt} e^{At} = Ae^{At}}$$

So $\frac{d\vec{x}}{dt} = Ae^{At}\vec{x_0} = A\vec{x}$. which shows $\vec{x}(t) = e^{At}\vec{x_0}$ is the solution to the initial value problem.

Properties of matrix exponential functions [Theorem 6.5.3]

Let A & B be $n\times n$ matrices. and t, τ be real or complex numbers.

Then

(a) $e^{A(t+\tau)} = e^{At}e^{A\tau}$

(b). $Ae^{At} = e^{At}A$

(c) $(e^{At})^{-1} = e^{-At}$

(d). $e^{(A+B)t} = e^{At}e^{Bt}$ if $AB = BA$.

Given matrix A, how to compute e^{At}?

Consider the initial value problem $\vec{x}'(t) = A\vec{x}$ $\vec{x}(0) = \vec{x}_0$

Find a fundamental set of solutions: $\vec{x}_1(t)\ \vec{x}_2(t) \cdots \vec{x}_n(t)$

Define the matrix $X(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix}$

$$e^{At} = X(t)X^{-1}(0)$$

Ex4. Find e^{At} for the system $\vec{x}'(t) = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} A\vec{x}$.

and compute $(e^{At})^{-1}$.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (1-\lambda)(3-\lambda)+1 \\ &= 3-\lambda-3\lambda+\lambda^2+1 \\ &= \lambda^2-4\lambda+4 \\ &= (\lambda-2)^2. \end{aligned}$$

$$\lambda = 2 \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Fundamental sol. $\vec{X}_1(t) = e^{\lambda t}\vec{v} = e^{2t}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}e^{2t}\\-e^{2t}\end{pmatrix}$

$\vec{X}_2(t) = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$

where $(A-\lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix}-1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

$w_1 + w_2 = -1$ Let $w_1 = 0$ $w_2 = -1$

$$\vec{w} = \begin{pmatrix}0\\-1\end{pmatrix}$$

$$\vec{X}_2(t) = te^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + e^{2t}\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}te^{2t}\\-te^{2t}-e^{2t}\end{pmatrix}$$

Then $X(t) = \begin{pmatrix}e^{2t} & te^{2t}\\-e^{2t} & -te^{2t}-e^{2t}\end{pmatrix}$ $X(0) = \begin{pmatrix}1 & 0\\-1 & -1\end{pmatrix}$

Compute $X^{-1}(0)$ $\left(\begin{array}{cc|cc}1 & 0 & 1 & 0\\-1 & -1 & 0 & 1\end{array}\right) \to \left(\begin{array}{cc|cc}1 & 0 & 1 & 0\\0 & -1 & 1 & 1\end{array}\right)$

$\to \left(\begin{array}{cc|cc}1 & 0 & 1 & 0\\0 & 1 & -1 & -1\end{array}\right)$ $X^{-1}(0) = \begin{pmatrix}1 & 0\\-1 & -1\end{pmatrix}$

$$e^{At} = X(t)X^{-1}(0) = \begin{pmatrix}e^{2t} & te^{2t}\\-e^{2t} & -te^{2t}-e^{2t}\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & -1\end{pmatrix} = \begin{pmatrix}e^{2t}-te^{2t} & -te^{2t}\\te^{2t} & te^{2t}+e^{2t}\end{pmatrix},$$

$$(e^{At})^{-1} = e^{-At} = \begin{pmatrix}e^{-2t}+te^{-2t} & te^{-2t}.\\-te^{-2t} & -te^{-2t}+e^{-2t}\end{pmatrix}$$

Section 6.6. Nonhomogeneous linear systems.

Nonhomogeneous system.

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \qquad \vec{x}(t_0) = \vec{x}_0$$

Homogeneous system

$$\vec{x}' = P(t)\vec{x}$$

Solution of the homogeneous system $\vec{x} = X(t)\vec{c}$

$$= \begin{bmatrix} \vec{x}_1(t) & \cdots & \vec{x}_n(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ form a fundamental sol of the homogeneous system.

<u>Method of variation parameters</u>

Let the solution of the non-homogeneous system be.

$$\vec{x} = \underbrace{X(t)}_{n\times n}\, \underbrace{\vec{u}(t)}_{n\times 1} = \begin{bmatrix} \vec{x}_1(t) & \cdots & \vec{x}_n(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

$$\vec{x}' = X'(t)\vec{u}(t) + X(t)\vec{u}'(t).$$

$$= P(t)X(t)\vec{u}(t) + X(t)\vec{u}'(t)$$

Notice $X'(t) = \begin{bmatrix} \vec{x}_1'(t) & \cdots & \vec{x}_n'(t) \end{bmatrix}$

$$= \begin{bmatrix} P(t)\vec{x}_1(t) & \cdots & P(t)\vec{x}_n(t) \end{bmatrix}$$

$$= P(t)X(t)$$

Let $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$

$$\Rightarrow P(t)X(t)\vec{u}(t) + X(t)\vec{u}'(t) = P(t)X(t)\vec{u}(t) + \vec{g}(t).$$

$$\Rightarrow X(t)\vec{u}'(t) = \vec{g}(t).$$

$$\vec{u}'(t) = X^{-1}(t)\,\vec{g}(t).$$

$$\Rightarrow \vec{u}(t) = \int_{t_0}^{t} X^{-1}(s)\,\vec{g}(s)\,ds.$$

General sol. of the nonhomogeneous system is.

$$\vec{x}(t) = \underbrace{X(t)\,\vec{c}}_{\substack{\text{solution of the}\\ \text{homogeneous system}}} + \underbrace{X(t)\int_{t_0}^{t} X^{-1}(s)\,\vec{g}(s)\,ds}_{\text{a particular solution}}.$$

Plug in the initial value $\vec{x}(t_0) = X(t_0)\vec{c} = \vec{x}_0 \Rightarrow \vec{c} = X^{-1}(t_0)\vec{x}_0$

Solution of the initial value problem.

$$\boxed{\vec{x}(t) = X(t)\,X^{-1}(t_0)\,\vec{x}_0 + X(t)\int_{t_0}^{t} X^{-1}(s)\,\vec{g}(s)\,ds}.$$

We can express $\vec{x}(t)$ in terms of fundamental matrices. $\Phi(t) = X(t)\,X(t_0)^{-1}$

$$\vec{x}(t) = X(t)X^{-1}(t_0)\,\vec{x}_0 + X(t)X^{-1}(t_0)\int_{t_0}^{t}\left(X(s)X^{-1}(t_0)\right)^{-1}\vec{g}(s)\,ds.$$

$$= \Phi(t)\,\vec{x}_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\,\vec{g}(s)\,ds.$$

In case of nonhomogeneous system with constant coefficients

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

Fundamental matrix $\Phi(t) = e^{At}$.

General sol. $\vec{x}(t) = e^{At}\vec{c} + e^{At}\int_{t_0}^{t} e^{-As}\,\vec{g}(s)\,ds.$

Plug in initial value.

$$\vec{x}(t_0) = e^{At_0}\vec{c} = \vec{x_0} \Rightarrow \vec{c} = (e^{At_0})^{-1}\vec{x_0} = e^{-At_0}\vec{x_0}$$

Solution to the initial value problem

$$\vec{x}(t) = e^{A(t-t_0)}\vec{x_0} + e^{At}\int_{t_0}^{t} e^{-As}\vec{g}(s)\,ds.$$

Ex1. $\vec{x}' = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} e^{-t} \\ t \end{pmatrix}.$

Given fundamental solutions $\vec{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{x}_2(t) = e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

$$X(t) = \begin{pmatrix} e^{-t} & -e^{t} \\ 0 & e^{t} \end{pmatrix}. \qquad \det X(t) = 1.$$

If $X(t) = \begin{bmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{bmatrix}$ then $X^{-1}(t) = \begin{bmatrix} X_{22}(t) & -X_{12}(t) \\ -X_{21}(t) & X_{11}(t) \end{bmatrix}\dfrac{1}{\det(X)}$

$$X^{-1}(t) = \begin{bmatrix} e^{t} & e^{t} \\ 0 & e^{-t} \end{bmatrix}$$

$$\vec{u}'(t) = X^{-1}(t)\vec{g}(t) = \begin{bmatrix} e^{t} & e^{t} \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} e^{-t} \\ t \end{bmatrix} = \begin{bmatrix} 1+te^{t} \\ te^{-t} \end{bmatrix}.$$

$u=e^{t}$ $dv=1$
$du=e^{t}dt$ $v=t$

$u=-e^{-t}$ $dv=dt$
$du=e^{-t}dt$ $v=t$

$$\int te^{t}dt = te^{t} - \int e^{t}dt = te^{t} - e^{t} \qquad \int te^{-t}dt = -te^{-t} + \int e^{-t}dt = -te^{-t} - e^{-t}$$

$$\vec{u}(t) = \begin{bmatrix} t + te^{t} - e^{t} \\ -te^{-t} - e^{-t} \end{bmatrix}$$

$$\vec{x}_p = X(t)\vec{u}(t) = \begin{bmatrix} e^{-t} & -e^{t} \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} t + te^{t} - e^{t} \\ -te^{-t} - e^{-t} \end{bmatrix}$$

$$e^{-t}(t + te^{t} - e^{t}) - e^{t}(-te^{-t} - e^{-t}) = te^{-t} + t - 1 + t + 1 = te^{-t} + 2t.$$

$$\vec{x}_p = \begin{bmatrix} te^{-t} + 2t \\ -t - 1 \end{bmatrix}$$

Genal sol $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} te^{-t} + 2t \\ -t - 1 \end{pmatrix}.$

Ex2. Solve $\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{x} + \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

Given fundamental solutions. $\vec{x}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ $\vec{x}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$.

$$X(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad \det X(t) = \cos^2 t + \sin^2 t = 1$$

$$X^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

$$\vec{u}'(t) = X^{-1}(t)\,\vec{g}(s) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\vec{u}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Particular sol. $\vec{x}_p(t) = X(t)\vec{u}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} t\cos t \\ -t\sin t \end{pmatrix}$

Genenal sol. $\vec{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} t\cos t \\ -t\sin t \end{pmatrix}.$

Initial value $\vec{x}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

Solution of the initial value problem $\vec{x}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} t\cos t \\ -t\sin t \end{pmatrix}$

Section 6.7 Defective matrices

Theorem 6.7.1. Consider the homogeneous system $\vec{x}' = A\vec{x}$

Suppose λ is an eigenvalue of A with algebraic multiplicity m.

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be linearly independent solutions of $(A-\lambda I)^m \vec{v} = \vec{0}$

Then

$$\vec{X}_1 = e^{\lambda t}\left[\vec{v}_1 + \frac{t}{1!}(A-\lambda I)\vec{v}_1 + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}_1\right]$$

$$\vec{X}_2 = e^{\lambda t}\left[\vec{v}_2 + \frac{t}{1!}(A-\lambda I)\vec{v}_2 + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}_2\right]$$

$$\vec{X}_3 = e^{\lambda t}\left[\vec{v}_3 + \frac{t}{1!}(A-\lambda I)\vec{v}_3 + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}_3\right]$$

$$\vdots$$

$$\vec{X}_m = e^{\lambda t}\left[\vec{v}_m + \frac{t}{1!}(A-\lambda I)\vec{v}_m + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}_m\right]$$

are linearly independent solutions of $\vec{x}' = A\vec{x}$.

Ex1. Find a fundamental set of solutions for

$$\vec{x}' = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}}_{A} \vec{x}.$$

Eigenvalues & eigenvectors of A

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (1-\lambda)(3-\lambda) - (-1) \\ &= 3 - 3\lambda - \lambda + \lambda^2 + 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2. \end{aligned}$$

$\lambda = 2$. Algebraic multiplicity $m=2$.

$$A-\lambda I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \qquad (A-\lambda I)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Let $(A-\lambda I)^2 \vec{v} = \vec{0}$

Two linearly independent solutions are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\vec{x}_1 = e^{\lambda t}(\vec{v}_1 + t(A-\lambda I)\vec{v}_1)$$

$$= e^{\lambda t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = e^{2t}\begin{pmatrix} 1-t \\ t \end{pmatrix}.$$

$$\vec{x}_2 = e^{\lambda t}(\vec{v}_2 + t(A-\lambda I)\vec{v}_2)$$

$$= e^{\lambda t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^{2t}\begin{pmatrix} -t \\ 1+t \end{pmatrix}.$$

Ex2. Find a fundamental set of solutions for.

$$\vec{x}' = \underbrace{\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}}_{A} \vec{x}$$

$$A-\lambda I = \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{bmatrix}$$

Compute $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ 0 & -\lambda-2 & -\lambda \end{vmatrix}$

$$= (4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ -\lambda-2 & -\lambda \end{vmatrix} - \begin{vmatrix} 6 & 6 \\ -\lambda-2 & -\lambda \end{vmatrix}$$

$$= (4-\lambda)[-\lambda(3-\lambda)+2(\lambda+2)] - [-6\lambda + 6(\lambda+2)]$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda^3 - 5\lambda^2 + 8\lambda - 4) = -(\lambda-1)(\lambda-2)^2.$$

$\lambda_1 = 1$ algebraic multiplicity 1.

Eigenvector
$$\begin{bmatrix} \cancel{3} & \cancel{6} & \cancel{6} \\ 1 & 2 & 2 \\ 0 & -3 & -1 \end{bmatrix}$$
$$\uparrow$$
$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}$$

$v_1 + 2v_2 + 2v_3 = 0.$ $\quad v_1 + 2v_2 - 6v_2 = v_1 - 4v_2 = 0 \quad v_1 = 4v_2.$

$3v_2 + v_3 = 0. \Rightarrow v_3 = -3v_2$

$$\vec{v}_1 = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$$

One solution $\vec{v}_1 = e^t \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$.

$\lambda_2 = 2$ algebraic multiplicity 2.

$$A - \lambda_2 I = \begin{pmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{pmatrix}$$

$$(A - \lambda_2 I)^2 = \begin{pmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{pmatrix}\begin{pmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{pmatrix} = \begin{pmatrix} 4 & -12 & 0 \\ 1 & -3 & 0 \\ -3 & 9 & 0 \end{pmatrix}$$

Solve $(A - \lambda_2 I)^2 \vec{v} = \vec{0}$ by elementary row operations.

$$\begin{pmatrix} 4 & -12 & 0 \\ 1 & -3 & 0 \\ -3 & 9 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 4 & -12 & 0 \\ -3 & 9 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad v_1 - 3v_2 = 0$$

$$\vec{v} = \begin{pmatrix} 3v_2 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Take two independent solutions $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Therefore.

$$\vec{X}_2(t) = e^{\lambda_2 t}\left[\vec{V}_2 + t(A-\lambda_2 I)\vec{V}_2\right]$$

$$= e^{2t}\left(\begin{bmatrix}3\\1\\0\end{bmatrix} + t\begin{bmatrix}2 & 6 & 6\\1 & 1 & 2\\-1 & -5 & -4\end{bmatrix}\begin{bmatrix}3\\1\\0\end{bmatrix}\right)$$

$$= e^{2t}\begin{pmatrix}3+12t\\1+4t\\-8t\end{pmatrix}$$

$$\vec{X}_3(t) = e^{\lambda_2 t}\left[\vec{V}_3 + t(A-\lambda_2 I)\vec{V}_3\right]$$

$$= e^{2t}\left(\begin{bmatrix}0\\0\\1\end{bmatrix} + t\begin{bmatrix}2 & 6 & 6\\1 & 1 & 2\\-1 & -5 & -4\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= e^{2t}\begin{pmatrix}6t\\2t\\1-4t\end{pmatrix}.$$

Ex 3. Find the general solution of.

$$\vec{x}\,' = \underbrace{\begin{pmatrix}-3 & -1 & -6\\-2 & -1 & -4\\1 & 0 & 1\end{pmatrix}}_{A}\vec{x}$$

$$A-\lambda I = \begin{pmatrix}-3-\lambda & -1 & -6\\-2 & -1-\lambda & -4\\1 & 0 & 1-\lambda\end{pmatrix}$$

$$\det(A-\lambda I) = (-1)^{3+1}\begin{vmatrix}-1 & -6\\-1-\lambda & -4\end{vmatrix} + (1-\lambda)\begin{vmatrix}-3-\lambda & -1\\-2 & -1-\lambda\end{vmatrix}$$

$$= -(\lambda^3+3\lambda^2+3\lambda+1) = -(\lambda+1)^3.$$

$\lambda = -1$ algebraic multiplicity 3.

$$A - \lambda I = \begin{pmatrix} -2 & -1 & -6 \\ -2 & 0 & -4 \\ 1 & 0 & 2 \end{pmatrix}.$$

$$(A-\lambda I)^2 = \begin{pmatrix} -2 & -1 & -6 \\ -2 & 0 & -4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & -1 & -6 \\ -2 & 0 & -4 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{pmatrix}.$$

$$(A-\lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Three linearly independent solutions of $(A-\lambda I)^3 \vec{v} = \vec{0}$.

$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x_1}(t) = e^{\lambda t}\left[\vec{v_1} + t(A-\lambda I)\vec{v_1} + \frac{t^2}{2}(A-\lambda I)^2\vec{v_1}\right]$$

$$= e^{-t}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 & -1 & -6 \\ -2 & 0 & -4 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = e^{-t}\begin{pmatrix} 1-2t \\ -2t \\ t \end{pmatrix}$$

$$\vec{x_2}(t) = e^{\lambda t}\left(\vec{v_2} + t(A-\lambda I)\vec{v_2} + \frac{t^2}{2}(A-\lambda I)^2\vec{v_2}\right)$$

$$= e^{-t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 & -1 & -6 \\ -2 & 0 & -4 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = e^{-t}\begin{pmatrix} -t+t^2 \\ 1+t^2 \\ -t^2/2 \end{pmatrix}$$

$$\vec{x_3}(t) = e^{-t}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 & -1 & -6 \\ -2 & 0 & -4 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

$$= e^{-t}\begin{pmatrix} -6t+2t^2 \\ -4t+2t^2 \\ 1+2t-t^2 \end{pmatrix}.$$

Theorem 6.7.1 complex eigenvalues.

Suppose. $u+vi$ is an eigenvalue of A with algebraic multiplicity m.

$u-vi$ " " " " " " m.

then $2m$ linearly independent, real-valued solutions of $\vec{x}' = A\vec{x}$ associated with λ are. $\{ \operatorname{Re} \vec{x}_1, \ldots, \operatorname{Re} \vec{x}_m \}$

and $\{ \operatorname{Im} \vec{x}_1, \ldots, \operatorname{Im} \vec{x}_m \}$.

Ex: Find a fundamental set of solutions of

$$y^{(4)} + 2y'' + y = 0.$$

Let $x_1 = y$ $\qquad x_1' = y' = x_2$

$x_2 = y'$ $\qquad x_2' = y'' = x_3$

$x_3 = y''$ $\qquad x_3' = y^{(3)} = x_4$

$x_4 = y^{(3)}$ $\qquad x_4' = y^{(4)} = -y - 2y'' = -x_1 - 2x_3$.

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}}_{A} \vec{x}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -1 & 0 & -2 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & -\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -2 & -\lambda \end{vmatrix}$$

Expand along the 1st row.

$$= -\lambda(-\lambda)\begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & 0 \\ -\lambda & 1 \end{vmatrix}.$$

$$= \lambda^2(\lambda^2+2) + 1 = \lambda^4 + 2\lambda^2 + 1 = (\lambda^2+1)^2 = (\lambda+i)^2(\lambda-i)^2.$$

Eigenvalue $\lambda_1 = i$ algebraic multiplicity $m = 2$.

$\lambda_2 = -i$ algebraic multiplicity $m = 2$.

Take $\lambda_1 = i$ Solve. $(A - \lambda_1 I)^2 \vec{v} = 0$.

$$A - \lambda_1 I = \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ -1 & 0 & -2 & -i \end{pmatrix}$$

$$(A-\lambda_1 I)^2 = \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ -1 & 0 & -2 & -i \end{pmatrix}\begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ -1 & 0 & -2 & -i \end{pmatrix} = \begin{pmatrix} -1 & -2i & 1 & 0 \\ 0 & -1 & -2i & 1 \\ -1 & 0 & -3 & -2i \\ 2i & -1 & 4i & -3 \end{pmatrix}$$

$$\begin{array}{c} \\ \\ -①+③ \\ 2i①+④ \end{array}\begin{pmatrix} -1 & -2i & 1 & 0 \\ 0 & -1 & -2i & 1 \\ -1 & 0 & -3 & -2i \\ 2i & -1 & 4i & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Elementary row operations.

$$\begin{array}{c} \\ \\ 2i②+③ \\ 3②+④ \end{array}\begin{pmatrix} -1 & -2i & 1 & 0 \\ 0 & -1 & -2i & 1 \\ 0 & 2i & -4 & -2i \\ 0 & 3 & 6i & -3 \end{pmatrix} \xrightarrow{-2i②+①} \begin{pmatrix} -1 & -2i & 1 & 0 \\ 0 & -1 & -2i & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\longrightarrow \begin{pmatrix} -1 & 0 & -3 & -2i \\ 0 & -1 & -2i & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-v_1 \qquad -3v_3 - 2i v_4 = 0. \quad \Rightarrow \quad v_1 = -3v_3 - 2i v_4$$

$$-v_2 - 2i v_3 + v_4 = 0 \quad \Rightarrow \quad v_2 = -2i v_3 + v_4$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} -3v_3 - 2i v_4 \\ -2i v_3 + v_4 \\ v_3 \\ v_4 \end{pmatrix} = v_3 \begin{pmatrix} -3 \\ -2i \\ 1 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} -2i \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Two independent solutions of $(A - \lambda_1 I)^2 \vec{v} = \vec{0}$ are.

$$\vec{v_1} = \begin{pmatrix} -3 \\ -2i \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v_2} = \begin{pmatrix} -2i \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{z}_1(t) = e^{\lambda_1 t} \left[\vec{v_1} + t (A - \lambda_1 I) \vec{v_1} \right]$$

$$= e^{it} \left(\begin{bmatrix} -3 \\ -2i \\ 1 \\ 0 \end{bmatrix} + t \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ -1 & 0 & -2 & -i \end{pmatrix} \begin{pmatrix} -3 \\ -2i \\ 1 \\ 0 \end{pmatrix} \right).$$

$$= e^{it} \begin{pmatrix} -3 + 3it - 2it \\ -2i - 2t + t \\ 1 - it \\ 3t - 2t \end{pmatrix} = e^{it} \begin{pmatrix} -3 + it \\ -2i - t \\ 1 - it \\ t \end{pmatrix}$$

$$= (\cos t + i \sin t) \left(\begin{bmatrix} -3 \\ -t \\ 1 \\ t \end{bmatrix} + i \begin{bmatrix} t \\ -2 \\ -t \\ 0 \end{bmatrix} \right)$$

$$= \cos t \begin{bmatrix} -3 \\ -t \\ 1 \\ t \end{bmatrix} - \sin t \begin{bmatrix} t \\ -2 \\ -t \\ 0 \end{bmatrix} + i \left(\sin t \begin{bmatrix} -3 \\ -t \\ 1 \\ t \end{bmatrix} + \cos t \begin{bmatrix} t \\ -2 \\ -t \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -3\cos t - t \sin t \\ -t\cos t + 2\sin t \\ \cos t + t \sin t \\ t \cos t \end{bmatrix} + i \begin{bmatrix} -3\sin t + t\cos t \\ -t\sin t - 2\cos t \\ \sin t - t\cos t \\ t \sin t \end{bmatrix}$$

Two real solutions.

$$\vec{x}_1(t) = \begin{bmatrix} -3\cos t - t\sin t \\ -t\cos t + 2\sin t \\ \cos t + t\sin t \\ t\cos t \end{bmatrix}, \qquad \vec{x}_2(t) = \begin{bmatrix} -3\sin t + t\cos t \\ -t\sin t - 2\cos t \\ \sin t - t\cos t \\ t\sin t \end{bmatrix}$$

$$\vec{z}_2(t) = e^{\lambda_1 t}\left[\vec{v}_2 + t(A - \lambda_1 I)\vec{v}_2\right]$$

$$= e^{it}\left(\begin{bmatrix} -2i \\ 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ -1 & 0 & -2 & -i \end{bmatrix}\begin{bmatrix} -2i \\ 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{it}\begin{pmatrix} -2i - t \\ 1 - it \\ t \\ 1 + 2it - it \end{pmatrix} = e^{it}\begin{pmatrix} -2i - t \\ 1 - it \\ t \\ 1 + it \end{pmatrix}$$

$$= (\cos t + i\sin t)\left(\begin{bmatrix} -t \\ 1 \\ t \\ 1 \end{bmatrix} + i\begin{bmatrix} -2 \\ -t \\ 0 \\ t \end{bmatrix}\right)$$

$$= \cos t\begin{bmatrix} -t \\ 1 \\ t \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} -2 \\ -t \\ 0 \\ t \end{bmatrix} + i\left(\cos t\begin{bmatrix} -2 \\ -t \\ 0 \\ t \end{bmatrix} + \sin t\begin{bmatrix} -t \\ 1 \\ t \\ 1 \end{bmatrix}\right)$$

$$= \underbrace{\begin{bmatrix} -t\cos t + 2\sin t \\ \cos t + t\sin t \\ t\cos t \\ \cos t - t\sin t \end{bmatrix}}_{\vec{x}_3(t)} + i\underbrace{\begin{bmatrix} -2\cos t - t\sin t \\ -t\cos t + \sin t \\ t\sin t \\ t\cos t + \sin t \end{bmatrix}}_{\vec{x}_4(t)}$$