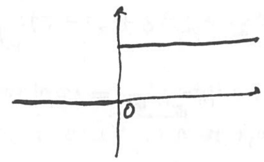
# Section 5.5 Discontinuous functions and periodic functions



## Discontinuous functions.

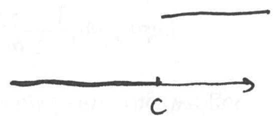
Def: [Unit step function or Heaviside function].

urn on 1 at 0 and Leave it on $\quad u(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0. \end{cases}$



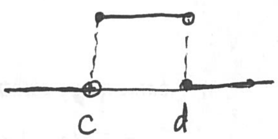

$u(t)$

urn on 1 at c and leave it on $\quad u_c(t) = \begin{cases} 0 & t<c. \\ 1 & t \geq c \end{cases}$




$u_c(t)$

[Indicator function] $\quad u_{cd} = \begin{cases} 0 & t<c \text{ or } t \geq d. \\ 1 & c \leq t < d. \end{cases}$




Suppose $c<d$

$= u_c(t) - u_d(t)$

- turn on 1 at $t=c$ and then turn it off at $t=d$.

Ex 1: $f(t) = \begin{cases} 0 & t<2. \\ 3 & 2 \leq t < 4 \\ 0. & t \geq 4 \end{cases}$ Write $f(t)$ in terms of heaviside functions.

- turn on 3 at $t=2$ and then. turn it off at $t=4$.

$f(t) = 3[u_2(t) - u_4(t)]$.

Ex 2: $f(t) = \begin{cases} t & 0<t<2. \\ 1 & 2 \leq t < 3. \\ e^{-2t} & t \geq 3 \end{cases}$ Express $f(t)$ in terms of heaviside functions.

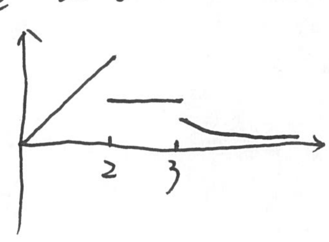
- turn on $t$ at $t=0$ and turn it off at $t=2$.
- turn on 1 at $t=2$ and turn it off at $t=3$
- turn on $e^{-2t}$ at $t=3$ and leave it on.

$f(t) = t\, u_{02}(t) + 1\, u_{23}(t) + e^{-2t} u_3(t)$.

$= t\,[u_0(t) - u_2(t)]$

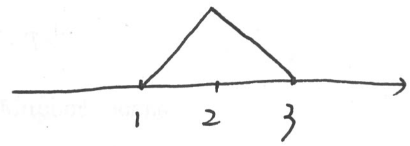
$+ 1\,[u_2(t) - u_3(t)] + e^{-2t} u_3(t)$.

$= t u_0(t) + (1-t) u_2(t) + (e^{-2t} - 1) u_3(t)$



Ex3. $f(t) = \begin{cases} 0 & t<1 \\ t-1 & 1\le t<2 \\ 3-t & 2\le t<3 \\ 0 & 3\le t \end{cases}$  Express $f(t)$ in terms of heaviside funs.

$f(t) = (t-1)\,u_{12}(t) + (3-t)\,u_{23}(t)$.




The Laplace transform of the heaviside function. $u_c(t)$ with $c \ge 0$.

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \quad \text{for } s>0$$

Proof: $\mathcal{L}\{u_c\} = \int_0^{+\infty} e^{-st}\,u_c(t)\,dt = \int_c^{\infty} e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_c^{\infty}$

$= 0 - \left(-\frac{1}{s}e^{-sc}\right) = \frac{1}{s}e^{-cs}$.

if $s>0$.

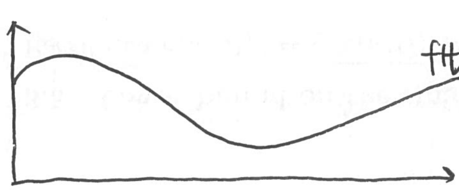
Then $\mathcal{L}\{u_{cd}(t)\} = \mathcal{L}\{u_c(t)\} - \mathcal{L}\{u_d(t)\}$.

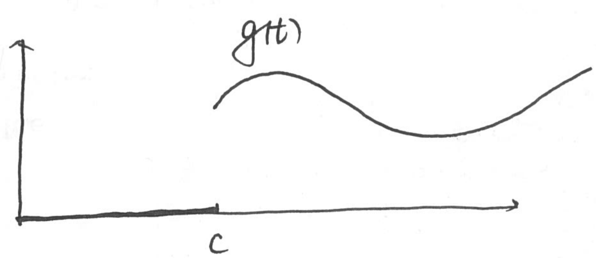
$= \dfrac{e^{-cs} - e^{-ds}}{s}$  $s>0$.

Shifted functions.

Given $f(t)$, defined $g(t) = \begin{cases} 0 & t<c. \\ f(t-c) & t\ge c. \end{cases}$  $g(t) = u_c(t)\,f(t-c)$.

$g(t)$ is a translation of $f$ a distance $c$ in the positive $t$ direction.



Laplace transform of time-shifted functions.

If $c \geq 0$, $\mathcal{L}\{f\}$ exists for $s > a \geq 0$

then $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\}$ for $s > a$.

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$

then $u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs} F(s)\}$.

Proof. $\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^{\infty} e^{-st} u_c(t) f(t-c)\, dt$.

$$= \int_c^{\infty} e^{-st} f(t-c)\, dt \quad \text{let } u = t-c.$$

$$= \int_0^{\infty} e^{-s(u+c)} f(u)\, du = e^{-cs} \int_0^{+\infty} e^{-su} f(u)\, du$$

$$= e^{-cs} \mathcal{L}\{f(t)\}.$$

Ex. 4. Find the Laplace transform of.

$$f(t) = \begin{cases} 0 & t < 1 \\ t-1 & 1 \leq t < 2. \\ 3-t & 2 \leq t < 3 \\ 0 & t \geq 3. \end{cases}$$

$$f(t) = (t-1) u_{12}(t) + (3-t) u_{23}(t)$$

$$= (t-1)[u_1(t) - u_2(t)] + (3-t)[u_2(t) - u_3(t)]$$

$$= (t-1) u_1(t) + (4-2t) u_2(t) + (t-3) u_3(t).$$

$$= (t-1) u_1(t) - 2(t-2) u_2(t) + (t-3) u_3(t).$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1) u_1(t)\} - 2\mathcal{L}\{(t-2) u_2(t)\} + \mathcal{L}\{(t-3) u_3(t)\}.$$

$$= e^{-s} \mathcal{L}\{t\} - 2e^{-2s} \mathcal{L}\{t\} + e^{-3s} \mathcal{L}\{t\}.$$

$$= \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}.$$

Ex 5. Find the inverse Laplace transform of

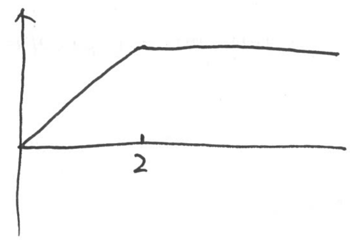
$$F(s) = \frac{1-e^{-2s}}{s^2}.$$

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

$$= t - u_2(t)\, g(t-2) \qquad g = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

$$= t - (t-2)\, u_2(t).$$

$$= \begin{cases} t & t<2 \\ 2. & t \geq 2. \end{cases}$$




Ex 6. Find the inverse Laplace transform of.

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$$F(s) = \frac{e^{-2s}}{s^2+s-2}.$$

$c=2.$

$$\mathcal{L}^{-1}\{F\} = u_2(t)\, g(t-2). \qquad g = \mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\}$$

$$= u_2(t)\left[\frac{1}{3}e^{t-2} - \frac{1}{3}e^{-2(t-2)}\right].$$

$$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}.$$

$$g(t) = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}.$$

# 5.6 Differential equations with discontinuous forcing.

Ex I. $y''(t) = u_c(t)$ $\quad y(0)=0 \quad y'(0)=0. \qquad c>0.$

acceleration

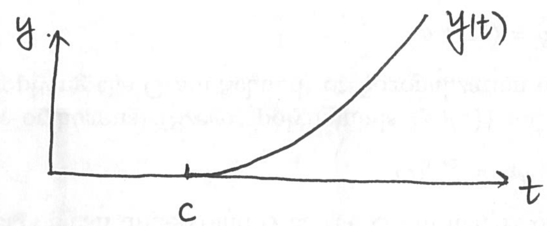
$$s^2Y - s\,y(0) - y'(0) = \mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$$

$$Y = \frac{e^{-cs}}{s^3}$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = u_c(t)\, g(t-c) \quad \text{where } g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{t^2}{2!} = \frac{t^2}{2}.$$

$$= \frac{(t-c)^2}{2} u_c(t)$$




EX 2. $y'' + 4y = g(t) \qquad y(0)=0 \quad y'(0)=0$

where
$$g(t) = \begin{cases} 0 & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1 & t \ge 10. \end{cases}$$

$$s^2Y - s\,y(0) - y'(0) + 4Y = \mathcal{L}\{g(t)\}.$$

Compute the right hand side.

$$g(t) = \frac{t-5}{5}[u_5(t) - u_{10}(t)] + u_{10}(t).$$

$$= \frac{1}{5}(t-5)u_5(t) + \frac{-t+10}{5} u_{10}(t)$$

$$= \frac{1}{5}(t-5)u_5(t) - \frac{t-10}{5} u_{10}(t).$$

$$\mathcal{L}\{g\} = \frac{1}{5}e^{-5s}\mathcal{L}\{t\} - \frac{1}{5}e^{-10s}\mathcal{L}\{t\}$$

$$= \frac{e^{-5s} - e^{-10s}}{5s^2}$$

Then $(s^2+4)Y = \dfrac{e^{-5s} - e^{-10s}}{5s^2}$

$$Y = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)}$$

$$y(t) = \frac{1}{5}\left[u_5(t)g(t-5) - u_{10}(t)g(t-10)\right]$$

where $g = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+4)}\right\}$

Partial fraction. $\dfrac{1}{s^2(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4}$

$$1 = As(s^2+4) + B(s^2+4) + s^2(Cs+D)$$
$$= (A+C)s^3 + (B+D)s^2 + 4As + 4B$$

Let $s=0$.

$4B=1 \quad B=\frac{1}{4}$

$4A=0 \quad A=0$

$A+C=0 \quad C=0$

$B+D=0. \quad D=-\frac{1}{4}$

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

Ex. Problem 6 in Section 5.6.

$$y'' + 3y' + 2y = u_2(t) \qquad y(0) = 6 \quad y'(0) = 6$$

$$s^2Y - s\,y(0) - y'(0) + 3\left(sY - y(0)\right) + 2Y = \mathcal{L}\{u_2(t)\}$$

$$s^2Y - 6s - 6 + 3sY - 18 + 2Y = \frac{e^{-2s}}{s}$$

$$(s^2+3s+2)Y = 6s+24 + \frac{e^{-2s}}{s}$$

$$Y = \frac{6s+24}{s^2+3s+2} + \frac{e^{-2s}}{s(s^2+3s+2)} = \frac{6s+24}{(s+1)(s+2)} + \frac{e^{-2s}}{s(s+1)(s+2)}$$

Compute $\mathcal{L}^{-1}\left\{\frac{6s+24}{(s+1)(s+2)}\right\}$ $\qquad \frac{6s+24}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

$$6s+24 = A(s+2) + B(s+1)$$

Let $s=-1$ $\qquad A = 18$

Let $s=-2$. $\qquad -B = 12 \qquad B = -12$.

$$\frac{6s+24}{(s+1)(s+2)} = \frac{18}{s+1} - \frac{12}{s+2}.$$

$$\mathcal{L}^{-1}\left\{\frac{6s+24}{(s+1)(s+2)}\right\} = 18e^{-t} - 12e^{-2t}.$$

Compute. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)(s+2)}\right\} = u_2(t)\,g(t-2)$ where $g = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$.

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}.$$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1).$$

$s=0 \qquad 2A = 1 \qquad A = \frac{1}{2}$

$s=-1 \qquad -B = 1 \qquad B = -1$

$s=-2 \qquad 2C = 1 \qquad C = \frac{1}{2}.$

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}.$$

8

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

Then. $$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)(s+2)}\right\} = u_2(t)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]$$

Finally $$y(t) = .18e^{-t} - .12e^{-2t} + u_2(t)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]$$

Compute Laplace transforms (Exercises)



① $\mathcal{L}\{e^{-2t}\sin 3t\}$ ② $\mathcal{L}\{e^{-2t}(t^2+1)^2\}$ ③ $\mathcal{L}\{t^2\sin bt\}$

④ $\mathcal{L}\{te^{at}\sin bt\}$

① $= \mathcal{L}\{\sin 3t\}_{s\to s+2} = \left.\dfrac{3}{s^2+3^2}\right|_{s\to s+2} = \dfrac{3}{(s+2)^2+9}$

② $= \mathcal{L}\{e^{-2t}(t^4+2t^2+1)\}$

$= \mathcal{L}\{t^4\}_{s\to s+2} + 2\mathcal{L}\{t^2\}_{s\to s+2} + \mathcal{L}\{1\}_{s\to s+2}$

$= \left.\dfrac{4!}{s^5}\right|_{s\to s+2} + 2\left.\dfrac{2!}{s^3}\right|_{s\to s+2} + \left.\dfrac{1}{s}\right|_{s\to s+2}$

$= \dfrac{24}{(s+2)^5} + \dfrac{4}{(s+2)^3} + \dfrac{1}{s+2}$

③ Let $F(s) = \mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2} = b(s^2+b^2)^{-1}$

$\mathcal{L}\{t^2\sin bt\} = (-1)^2 F^{(2)}(s) = \dfrac{d^2F}{ds^2}$

$\dfrac{dF}{ds} = -b(s^2+b^2)^{-2}\cdot 2s = -\dfrac{2bs}{(s^2+b^2)^2}$

$\dfrac{d^2F}{ds^2} = -\dfrac{2b(s^2+b^2)^2 - 2bs\cdot 2(s^2+b^2)\cdot 2s}{(s^2+b^2)^4} = -\dfrac{(s^2+b^2)[2bs^2+2b^3-8bs^2]}{(s^2+b^2)^4}$

④ $= \mathcal{L}\{t\sin bt\}_{s \to s-a} = \dfrac{2b(s-a)}{[(s-a)^2+b^2]^2}.$

Let $F(s) = \mathcal{L}\{\sin bt\}$.

Then $\mathcal{L}\{t\sin bt\} = -\dfrac{dF}{ds} = \dfrac{2bs}{(s^2+b^2)^2}$