Section. 5.1 The Laplace transform.

Def 5.1.1 [The Laplace transform]

Let f be a function on $[0,\infty)$. The Laplace transform of f is

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt = \lim_{A \to \infty} \int_0^A e^{-st} f(t)\, dt.$$

for s such that the integral converges.

Notation. $\mathcal{L}\{f\} = F \qquad \mathcal{L}\{x\} = X \qquad \mathcal{L}\{y\} = Y.$

Ex.1 Let $f(t) = 1$, $t \geq 0$.

$$\mathcal{L}\{1\}(s) = \int_0^{\infty} e^{-st}\, dt = -\frac{e^{-st}}{s}\Bigg|_0^{\infty} = 0 + \frac{e^0}{s} = \frac{1}{s} \qquad s>0$$

Ex2. Let $f(t) = e^{at}$, $t \geq 0$. a is a real number.

$$\mathcal{L}\{e^{at}\}(s) = \int_0^{\infty} e^{-st} e^{at}\, dt = \int_0^{+\infty} e^{-(s-a)t}\, dt.$$

$$= -\frac{e^{-(s-a)t}}{s-a}\Bigg|_0^{\infty} = \frac{1}{s-a} \qquad s>a.$$

Ex3. $f(t) = e^{(a+bi)t}$, $t \geq 0$

$$\mathcal{L}\{e^{(a+bi)t}\} = \int_0^{+\infty} e^{-st} e^{(a+bi)t}\, dt = \int_0^{\infty} e^{-(s-a-bi)t}\, dt.$$

$$= -\frac{e^{-(s-a-bi)t}}{s-a-bi}\Bigg|_0^{\infty} = \frac{1}{s-a-bi} \qquad s>a.$$

Linearity.

$\mathcal{L}: f \longrightarrow F$ as an operator.

Claim: The Laplace transform is a linear operator.

Theorem 5.1.2. Suppose. $\mathcal{L}\{f_1\}(s)$ exists for $s > a_1$
$\mathcal{L}\{f_2\}(s)$ exists for $s > a_2$

Let c_1 and c_2 be real or complex numbers. Then, for $s > \max(a$

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$$

Ex 4. Find the Laplace transform of $f(t) = \sin at \quad t \geq 0$

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i} \qquad e^{iat} = \cos at + i \sin at$$
$$e^{-iat} = \cos at - i \sin at$$

$$\sin at = \frac{1}{2i} e^{iat} - \frac{1}{2i} e^{-iat}$$

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s - ai} \qquad \mathcal{L}\{e^{-iat}\} = \frac{1}{s + ai}$$

$$\mathcal{L}\{\sin at\} = \frac{1}{2i(s-ai)} - \frac{1}{2i(s+ai)} = \frac{s+ai-(s-ai)}{2i(s+ai)(s-ai)}$$

$$= \frac{2ai}{2i(s^2+a^2)} = \frac{a}{s^2+a^2}.$$

Ex 5. $f(t) = 2 + 5e^{-2t} - 3\sin 4t \quad t \geq 0.$

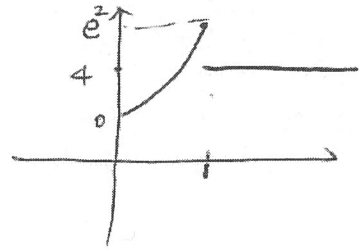
$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$$

$$\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16}$$

$$\mathcal{L}\{2 + 5e^{-2t} - 3\sin 4t\} = \frac{2}{s} + \frac{5}{s+2} - \frac{12}{s^2+16}.$$

Piecewise continuous functions.

$$f(t) = \begin{cases} e^{2t} & 0 \le t < 1 \\ 4 & 1 \le t \end{cases}$$



$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} e^{2t}\,dt + \int_1^\infty 4 e^{-st}\,dt.$$

$$= \int_0^1 e^{-(s-2)t}\,dt + 4\int_1^\infty e^{-st}\,dt$$

$$= -\frac{e^{-(s-2)t}}{s-2}\Big|_0^1 + 4\,\frac{e^{-st}}{-s}\Big|_1^\infty.$$

$$= -\frac{e^{-(s-2)}}{s-2} + \frac{1}{s-2.} + \frac{4e^{-s}}{s} \qquad s>0.$$

$$= \frac{1}{s-2} - \frac{e^{-(s-2)}}{s-2.} + \frac{4}{s}e^{-s} \qquad s>0.\ s\neq 2.$$

$$\mathcal{L}\{f\}(2) = 1 + \frac{4}{2}e^{-2} = 1+2e^{-2}.$$

When does the Laplace transform exist?

Def. [Functions of exponential order]

A function $f(t)$ is of exponential order. (as $t \to +\infty$) if there exist real constants. $M \ge 0$, $K>0$ and a such that.

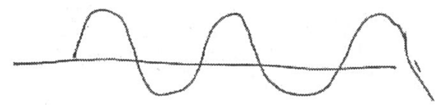
$$|f(t)| \le Ke^{at}.$$

when $t \ge M$.

Ex. Determine if the following functions are of exponential order.

(a) $f(t) = \cos at$

$|\cos at| \leq 1 = 1 \cdot e^{0 \cdot t}$



$K=1 \quad a=0 \quad M=0.$

(b). $f(t) = t^2$

Let $a > 0$.

$$\lim_{t \to \infty} \frac{t^2}{e^{at}} = \lim_{t \to \infty} \frac{2t}{a e^{at}} = \lim_{t \to \infty} \frac{2}{a^2 e^{at}} = 0.$$

From the definition of limit, there exists $M>0$ such that

$$\frac{t^2}{e^{at}} \leq 1 \qquad \text{when } t \geq M.$$

$$t^2 \leq 1 e^{at} \quad \text{when } t \geq M.$$

$\boxed{k=1}$

(c). $f(t) = e^{t^2}$.

$$\lim_{t \to \infty} \frac{e^{t^2}}{e^{at}} = \lim_{t \to \infty} e^{t(t-a)} = \infty.$$

No matter what a is, $e^{t^2} >> e^{at}$, $f(t) = e^{t^2}$ is not of exponential order.

Theorem 5.1.6.

If f is of exponential order: $|f(t)| \leq k e^{at}$ $t \geq M$.

Then the Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > a$.

Section 5.2.

Laplace transform of $e^{ct}f(t)$.

Theorem 5.2.1.
If $\mathcal{L}\{f\}(s)$ exists for $s>a$ and if c is a constant, then
$$\mathcal{L}\{e^{ct}f(t)\} = \mathcal{L}\{f\}(s-c) \text{ for } s>a+c.$$

Proof. $\mathcal{L}\{e^{ct}f(t)\}(s) = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)$

~~$u=s-c$.~~ $\quad = \mathcal{L}\{f\}(s-c)$. when $s-c>a$

$\cancel{= \int_0^{+\infty} e^{-ut}}$ $\quad s>a+c$. □

Ex. Find the Laplace transform. $g(t) = e^{-2t}\sin 4t$ and determine where it is valid.

$$\mathcal{L}\{e^{-2t}\sin 4t\} = \mathcal{L}\{\sin 4t\}(s+2) = \frac{4}{(s+2)^2+4^2} \qquad \begin{matrix} s+2>0 \\ s>-2. \end{matrix}$$
$c=-2$. $f(t)=\sin 4t$

Laplace transform of $f'(t)$.

Theorem 5.2.2. Suppose f and f' are of exponential order.

$|f(t)| \le K_1 e^{at}$ $t\ge M_1$ then $\mathcal{L}\{f\}(s)$ exists $s>a$.

$|f'(t)| \le K_2 e^{at}$ $t\ge M_2$ then $\mathcal{L}\{f\}(s)$ exist $s>a$.

Then $$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

$u=e^{-st}$ $\quad dv=f'(t)\,dt$

Proof. $\mathcal{L}\{f'\}(s) = \int_0^{+\infty} e^{-st}f'(t)\,dt$. $\quad du=-se^{-st}$ $\quad v=f(t)$

$$= e^{-st}f(t)\Big|_0^{+\infty} + s\int_0^{+\infty} e^{-st}f(t)\,dt$$

$$\lim_{t\to\infty} e^{-st}f(t) \le \lim_{t\to\infty} e^{-st}e^{at} = \lim_{t\to\infty} e^{-(s-a)t} = 0 \quad \text{for } s>a.$$

then $\underline{\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\} - f(0)}$

$$\begin{aligned}\mathcal{L}\{f''\} &= s\,\mathcal{L}\{f'\} - f'(0) \\ &= s\left(s\,\mathcal{L}\{f\} - f(0)\right) - f'(0) \\ &= s^2\mathcal{L}\{f\} - s f(0) - f'(0)\end{aligned}$$

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}($$

Use Laplace transform to solve DE.

Ex. Find the Laplace transform of

$$y'' + 2y' + 5y = e^{-t} \qquad y(0) = 1 \quad y'(0) = -3$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - s f(0) - f'(0)$$

$$\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - f(0)$$

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 2s\,\mathcal{L}\{y\} - 2y(0) + 5\mathcal{L}\{y\} = \frac{1}{s+1}$$

$$(s^2+2s+5)\,\mathcal{L}\{y\} - y'(0) - (s+2)\,y(0) = \frac{1}{s+1}$$

$$(s^2+2s+5)\,\mathcal{L}\{y\} + 3 - (s+2) = \frac{1}{s+1}$$

$$(s^2+2s+5)\,\mathcal{L}\{y\} = \frac{1}{s+1} + (s-1) = \frac{1+s^2-1}{s+1} = \frac{s}{s-}$$

$$\mathcal{L}\{y\} = \frac{s^2}{(s+1)(s^2+2s+5)}$$

Ex. Find the Laplace transform of.

$$\frac{d^4y}{dt^4} - y = 0$$

$y(0)=0 \quad y'(0)=0 \quad y''(0)=0 \quad y^{(3)}(0)=1$

$$\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} - \mathcal{L}\{y\} = 0.$$

$$s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - s\,y''(0) - y^{(3)}(0) - \mathcal{L}\{y\} = 0$$

Let $Y = \mathcal{L}\{y\}$

$$s^4Y - 1 - Y = 0.$$

$$Y(s) = \frac{1}{s^4-1}$$

Laplace transform of $t^n f(t)$.

Theorem 5.2.4. Suppose $f(t)$ is exponential order with a:

$|f(t)| \le ke^{at}$ for $t \ge M$. Let $F = \mathcal{L}\{f\}$

Then $\mathcal{L}\{t^n f\} = (-1)^n F^{(n)}(s) \quad s > a.$

Proof: $F^{(n)}(s) = \frac{d^n}{ds^n}\int_0^{+\infty} e^{-st} f(t)\,dt = \int_0^{+\infty} \frac{\partial^n}{\partial s^n}(e^{-st}) f(t)\,dt.$

$= \int_0^{+\infty} (-t)^n e^{-st} f(t)\,dt = (-1)^n \int_0^{+\infty} e^{-st}\, t^n f(t)$

$= (-1)^n \mathcal{L}\{t^n f\}(s).$

Corollary 5.2.5.

For any integer $n \geq 0$.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad s>0$$

Proof. $f(t)=1 \qquad F(s)=\mathcal{L}\{f\}(s)=\frac{1}{s}$

$$\mathcal{L}\{t^n\} = (-1)^n F^{(n)}(s) = (-1)^n (-1)^n \frac{n!}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$F(s)=\frac{1}{s}$$

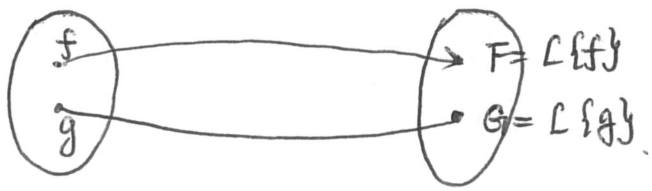
$$F'(s)=(-1)\frac{1}{s^2}$$

$$F''(s)=(-1)(-2)\frac{1}{s^3}$$

$$\vdots$$

$$F^{(n)}(s)=(-1)^n n! \frac{1}{s^{n+1}}$$

Section 5.3 The inverse Laplace transform.



Theorem 5.3.1

If $f(t)$ & $g(t)$ are piecewise continuous and of exponential order on $[0,\infty)$

If $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then $f(t) = g(t)$ at all points where both f and g are continuous. In particular, if f & g are continuous on $[0,\infty)$, then $f(t) = g(t)$ for all $t \geq 0$.

Eg: $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$ $\qquad g(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s) = (1-e^{-s})/s$$

Def. [Inverse Laplace transform].

If $\mathcal{L}\{f\} = F$, then we call f the inverse Laplace transform of F:

$$f = \mathcal{L}^{-1}\{F\}$$

Ex1. (a).

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = \sin 4t \quad \text{as} \quad \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at \quad a=4$$

(b) $\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\} = e^{-2t}t^3$

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \begin{matrix} a=-2 \\ n=3. \end{matrix}$$

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or $\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\Big|_{s\to s+2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = e^{-2t}t^3$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad n=3.$$

(c) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$

$= \mathcal{L}^{-1}\left\{ \frac{s}{s^2+2^2}\Big|_{s\to s+1} \right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = e^{-t}\cos 2t.$

Notice $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ $\quad a=2$

Linearity. Theorem 5.3.3

$\mathcal{L}^{-1}\{c_1F_1+c_2F_2\} = c_1\mathcal{L}^{-1}\{F_1\} + c_2\mathcal{L}^{-1}\{F_2\}$

Ex. $\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^4} + \frac{3}{s^2+16} + \frac{5(s+1)}{s^2+2s+5}\right\}$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\}$

$= \frac{1}{3}e^{-2t}t^3 + \frac{3}{4}\sin 4t + 5e^{-t}\cos 2t.$

Ex. $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2-4s+20}\right\}$

$\frac{3s+1}{s^2-4s+20} = \frac{3s+1}{(s-2)^2+4^2} = \frac{3(s-2)+7}{(s-2)^2+4^2} = 3\frac{s-2}{(s-2)^2+4^2} + \frac{7}{4}\frac{4}{(s-2)^2+4^2}$

$= 3\mathcal{L}^{-1}\left\{\overset{a=2\ b=4}{\frac{s-2}{(s-2)^2+4^2}}\right\} + \frac{7}{4}\mathcal{L}^{-1}\left\{\overset{a=2\ b=4}{\frac{4}{(s-2)^2+4^2}}\right\}$

$= 3e^{2t}\cos 4t + \frac{7}{4}e^{2t}\sin 4t.$

$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$ $\qquad \mathcal{L}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin bt.$

Partial fraction.

$$F(s) = \frac{P(s)}{Q(s)}. \qquad \text{eg:} \quad F(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

$\deg P < \deg Q$.

Partial fraction.

$$\frac{1}{s^4-1} = \frac{1}{(s+1)(s-1)(s^2+1)} = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s^2+1}.$$

Then $\mathcal{L}^{-1}\left\{\frac{1}{s^4-1}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= \frac{1}{4}e^t - \frac{1}{4}e^{-t} - \frac{1}{2}\sin t$

1. Nonrepeated linear factors.

$$Q(s) = (s-s_1)(s-s_2)\cdots(s-s_n)$$

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_n}{s-s_n}.$$

2. Repeated linear factors.

$$Q(s) = (s-s_j)^k \cdots$$

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_{j1}}{s-s_j} + \frac{a_{j2}}{(s-s_j)^2} + \cdots + \frac{a_{jk}}{(s-s_j)^k} + \cdots$$

3. Quadratic factors with complex roots $\mu+i\nu$ $\quad \mu-i\nu$.

if $Q(s) = [s-(\mu+i\nu)][s-(\mu-i\nu)] = (s-\mu)^2+\nu^2$.

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a(s-\mu)+b\nu}{(s-\mu)^2+\nu^2}.$$

if $Q(s) = [(s-\mu)^2+\nu^2]^2$.

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_1(s-\mu)+b_1\nu}{(s-\mu)^2+\nu^2} + \frac{a_2(s-\mu)+b_2\nu}{[(s-\mu)^2+\nu^2]^2}$$

Ex. $\mathcal{L}^{-1}\left\{ \frac{s-2}{s^2+4s-5} \right\}$

1. Write down the right form

$$\frac{s-2}{s^2+4s-5} = \frac{s-2}{(s+5)(s-1)} = \frac{a}{s-1} + \frac{b}{s+5}$$

2. Compute a & b

$$\frac{s-2}{(s+5)(s-1)} = \frac{a(s+5)+b(s-1)}{(s+5)(s-1)}$$

Or $(s-2) = a(s+5) + b(s-1)$

Let $s=1$ $\quad 6a=-1$

Let $s=-5$ $\quad -6b=-7$

$$s-2 = (a+b)s + 5a - b$$

$a+b=1$

$5a-b=-2$

$6a=-1 \quad a=-\frac{1}{6}$

$b=1-a=\frac{7}{6}$

$$\mathcal{L}^{-1}\left\{ \frac{s-2}{s^2+4s-5} \right\} = \mathcal{L}^{-1}\left\{ -\frac{1}{6}\frac{1}{s-1} + \frac{7}{6}\frac{1}{s+5} \right\}$$

$$= -\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{7}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = -\frac{1}{6}e^{t} + \frac{7}{6}e^{-5t}$$

Notice $\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

Ex. $\mathcal{L}^{-1}\left\{ \frac{s^2+20s+31}{(s+2)^2(s-3)} \right\}$

Or Let $s=3$ $\quad 25c=100 \quad c=4$

Let $s=-2$. $\quad -5b=-5 \quad b=1$

$$\frac{s^2+20s+31}{(s+2)^2(s-3)} = \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{c}{s-3}$$

$$\begin{aligned} s^2+20s+31 &= a(s+2)(s-3) + b(s-3) + c(s+2)^2 \\ &= a(s^2-s-6) + b(s-3) + c(s^2+4s+4) \\ &= (a+c)s^2 + (-a+b+4c)s + (-6a-3b+4c) \end{aligned}$$

$a+c=1$

$-a+b+4c=20$

$-6a-3b+4c=31$

$a=-3 \quad b=1 \quad c=4$

or $s=3$ $\quad 25c = 3^2+60+31 = 100$

$c=4$

$s=-2$. $\quad -5b = 4-40+31 = -5$

$b=1$

$$\mathcal{L}^{-1}\left\{\frac{s^2+20s+31}{(s+2)^2(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-3}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s-3}\right\}$$

$$= -3e^{-2t} + e^{-2t}t + 4e^{3t}.$$

as $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $\quad \mathcal{L}\{t\} = \frac{1}{s^2}$ $\quad \mathcal{L}\{e^{at}t\} = \frac{1}{(s-a)^2}$.

Ex. $\mathcal{L}^{-1}\left\{\frac{3s^2+2s+2}{s^3}\right\}$.

$$\frac{3s^2+2s+2}{s^3} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s^3}$$

$3s^2+2s+2 = as^2+bs+c$. $\quad a=3 \quad b=2 \quad c=2$.

$$\mathcal{L}^{-1}\left\{\frac{3s^2+2s+2}{s^3}\right\} = 3\overset{n=0}{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}} + 2\overset{n=1}{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}} + 2\overset{n=2}{\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}}$$

Notice. $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$. $\quad \mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

$$= 3 + 2t + 2\frac{t^2}{2!}$$

$$= 3 + 2t + t^2.$$

Ex. $\mathcal{L}^{-1}\left\{\frac{-3s^2-14s+32}{(s+4)(s^2+4)}\right\}$

$$\frac{-3s^2-14s+32}{(s+4)(\underbrace{s^2+4}_{4=2^2})} = \frac{a}{s+4} + \frac{bs+c}{s^2+4} = \frac{a}{s+4} + \frac{bs+c}{s^2+4}$$

$$-3s^2-14s+32 = a(s^2+4) + (bs+c)(s+4)$$
$$= (a+b)s^2 + (4b+c)s + 4a+4c$$

Let $s=-4$ $\quad 20a = -3\times16 - 14\times(-4) + 32 = -48+56+32 = 40$

$a=2$

$a+b=-3 \quad b=-5$

$4a+4c=32 \quad c = 8-a = 6$

$$\frac{-3s^2-14s+32}{(s+4)(s^2+4)} = \frac{2}{s+4} + \frac{-5s+6}{s^2+4}$$
$$= \frac{2}{s+4} + (-5)\frac{s}{s^2+4} + 3\cdot\frac{2}{s^2+4}$$

Answer $= 2e^{-4t} - 5\cos 2t + 3\sin 2t$

Ex. $\mathcal{L}^{-1}\left\{\dfrac{s^3-2s^2-6s-6}{(s^2+2s+2)\,s^2}\right\}$.

$$\frac{s^3-2s^2-6s-6}{(s^2+2s+2)\,s^2} = \frac{as+b}{s^2+2s+2} + \frac{c}{s} + \frac{d}{s^2}.$$

$$s^3-2s^2-6s-6 = s^2(as+b) + cs(s^2+2s+2) + d(s^2+2s+2)$$

$$= (a+c)s^3 + (b+2c+d)s^2 + (2c+2d)s + 2d$$

Let $s=0$. $\quad 2d=-6 \quad d=-3.$

$2c+2d=-6. \quad 2c=-6-2d=0 \quad c=0.$

$a+c=1 \quad a=1$

$b+2c+d=-2. \quad b=-2-d=1$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{s^2}\right\}.$$

$$= e^{-t}\cos t - 3t.$$

Compute
Not required $\mathcal{L}^{-1}\left\{\frac{s^2+3}{(s^2+2s+2)^2}\right\}$.

$$\frac{s^2+3}{[(s+1)^2+1]^2} = \frac{as+b}{(s+1)^2+1} + \frac{cs+d}{[(s+1)^2+1]^2}.$$

$$s^2+3 = (as+b)(s^2+2s+2) + cs+d.$$
$$= as^3 + (2a+b)s^2 + (2a+2b+c)s + 2b+d.$$

$a=0.$ $\quad 2a+b=1 \quad b=1 \quad 2a+2b+c=0 \quad c=-2b=-2.$

$d = 3-2b = 1.$

$$= \frac{1}{(s+1)^2+1} + \frac{-2s+1}{[(s+1)^2+1]^2}.$$

$$= \frac{1}{(s+1)^2+1} + \frac{-2(s+1)+2+1}{[(s+1)^2+1]^2}.$$

$$= \frac{1}{(s+1)^2+1} - 2\underset{??}{\frac{s+1}{[(s+1)^2+1]^2}} + 3\underset{??}{\frac{1}{[(s+1)^2+1]^2}}.$$

$$\mathcal{L}^{-1}\left\{\frac{(s+1)}{[(s+1)^2+1]^2}\right\} = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{s}{[s^2+1)^2}\right\}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{[(s+1)^2+1]^2}\right\} = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}.$$

How to compute $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$ & $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$. See Page. 22.

Then $y(t) = e^{-t}\sin t - 2e^{-t}\cdot\frac{1}{2}t\sin t + 3e^{-t}\cdot\frac{1}{2}(\sin t - t\cos t).$

$= \frac{5}{2}e^{-t}\sin t - te^{-t}\sin t - \frac{3}{2}te^{-t}\cos t.$

5.4 Solve. DE with Laplace transform.

Ex. $y' + 2y = \sin 4t \qquad y(0) = 1$

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\sin 4t\} \qquad \text{Let } Y = \mathcal{L}\{y\}$$

$$sY - y(0) + 2Y = \frac{4}{s^2+4^2}$$

$$(s+2)Y = \frac{4}{s^2+4^2} + 1 = \frac{s^2+20}{s^2+16}.$$

$$Y = \frac{s^2+20}{(s+2)(s^2+16)} = \frac{a}{s+2} + \frac{bs+c}{s^2+16}.$$

$$s^2+20 = a(s^2+16) + (bs+c)(s+2).$$
$$= (a+b)s^2 + (2b+c)s + 16a + 2c.$$

Let $s=-2$. $\quad 20a = 24 \quad a = \frac{6}{5}.$

$a+b=1 \quad b = -\frac{1}{5}.$

$2b+c=0 \quad c = -2b = \frac{2}{5}$

$$= \frac{6}{5}\frac{1}{s+2} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2+16}.$$

$$= \frac{6}{5}\frac{1}{s+2} - \frac{1}{5}\frac{s}{s^2+16} + \frac{2}{5\cdot 4}\frac{4}{s^2+4^2}.$$

$$y(t) = \frac{6}{5}e^{-2t} - \frac{1}{5}\cos 4t + \frac{1}{10}\sin 4t$$

Section 5.4.

Ex. Find solutions of

$$y'' + y = e^{-t}\cos 2t \qquad y(0) = 2 \quad y'(0) = 1$$

Apply the Laplace transform. , Let $\mathcal{L}\{y\} = Y$.

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{s+1}{(s+1)^2+4}$$

$$(s^2+1)\,Y(s) - 2s - 1 = \frac{s+1}{(s+1)^2+4}$$

$$(s^2+1)\,Y(s) = \frac{s+1}{(s+1)^2+4} + 2s+1$$

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{s+1}{(s^2+1)[(s+1)^2+4]}$$

Compute the inverse Laplace transform.

$$\frac{2s+1}{s^2+1} = 2\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\frac{s+1}{(s^2+1)[(s+1)^2+4]} = \frac{as+b}{s^2+1} + \frac{cs+d}{(s+1)^2+4}$$

Compute a, b, c, d.

$$\begin{aligned}
s+1 &= (as+b)[(s+1)^2+4] + (cs+d)(s^2+1)\\
&= (as+b)(s^2+2s+5) + (cs+d)(s^2+1)\\
&= as^3+2as^2+5as+bs^2+2bs+5b+cs^3+cs+ds^2+d\\
&= (a+c)s^3+(2a+b+d)s^2+(5a+2b+c)s+5b+d
\end{aligned}$$

$$\begin{aligned}
a \quad + c \quad\quad &= 0 \quad \Rightarrow c = -a\\
2a + b \quad + d &= 0\\
5a+2b+c \quad\quad &= 1\\
5b \quad + d &= 1 \quad \Rightarrow d = 1-5b
\end{aligned}$$

$$\begin{aligned}
2a+b \quad 1-5b &= 0 \Rightarrow 2a-4b = -1\\
5a+2b-a \quad &= 1 \Rightarrow 4a+2b = 1
\end{aligned}$$

$$\begin{aligned} 2a-4b&=-1 \\ 8a+4b&=2. \end{aligned} \Rightarrow \quad 10a=1 \quad a=\frac{1}{10}. \quad 4b=2a+1=\frac{12}{10} \quad b=\frac{12}{10\cdot 4}=\frac{3}{10}$$

$$a=\frac{1}{10} \quad b=\frac{3}{10} \quad c=-\frac{1}{10}. \quad d=1-5\cdot\frac{3}{10}=1-\frac{15}{10}=-\frac{5}{10}=-\frac{1}{2}.$$

$$\frac{s+1}{(s^2+1)[(s+1)^2+4]}=\frac{\frac{1}{10}s+\frac{3}{10}}{s^2+1}+\underbrace{\frac{-\frac{1}{10}s-\frac{1}{2}}{(s+1)^2+4}}=\frac{-\frac{1}{10}(s+1)+\frac{1}{10}-\frac{1}{2}}{(s+1)^2+4}$$

$$=\frac{1}{10}\frac{s}{s^2+1}+\frac{3}{10}\frac{1}{s^2+1}-\frac{1}{10}\frac{s+1}{(s+1)^2+4}-\frac{4}{10}\frac{1}{(s+1)^2+4}$$

$$=\frac{1}{10}\frac{s}{s^2+1}+\frac{3}{10}\frac{1}{s^2+1}-\frac{1}{10}\frac{s+1}{(s+1)^2+4}-\frac{1}{5}\frac{2}{(s+1)^2+4}$$

$$y(t)=2\cos t+\sin t.+\frac{1}{10}\cos t+\frac{3}{10}\sin t-\frac{1}{10}e^{-t}\cos 2t-\frac{1}{5}10^{-t}\sin 2t$$

$$=\frac{21}{10}\cos t+\frac{13}{10}\sin t-\frac{1}{10}e^{-t}\cos 2t-\frac{1}{5}e^{-t}\sin 2t.$$

Ex. $y^{(4)}+2y''+y=0.$ $\quad y(0)=1 \quad y'(0)=-1 \quad y''(0)=0 \quad y^{(3)}(0)=2.$

Apply the Laplace transform.

$$s^4Y-s^3y(0)-s^2y'(0)-sy''(0)-y^{(3)}(0)+2\left(s^2Y-sy(0)-y'(0)\right)$$

$$+Y=0.$$

$$(s^4+2s^2+1)Y\underbrace{-s^3+s^2-2-2s+2}_{-s^3+s^2-2s.}=0.$$

$$Y=\frac{s^3-s^2+2s}{s^4+2s^2+1}=\frac{s^3-s^2+2s.}{(s^2+1)^2}$$

$$= \frac{as+b}{s^2+1} + \frac{cs+d}{(s^2+1)^2}.$$

$$s^3 - s^2 + 2s = (as+b)(s^2+1) + cs + d.$$
$$= as^3 + as + bs^2 + b + cs + d$$
$$= as^3 + bs^2 + (a+c)s + b + d.$$

$a = 1$ $\quad a + c = 2 \quad c = 2 - a = 1$

$b = -1$ $\quad b + d = 0. \quad d = -b = 1$

Not required

$$Y(s) = \frac{s-1}{s^2+1} + \frac{s+1}{(s^2+1)^2}.$$

see next page.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\}$$

$$= \cos t - \sin t + \tfrac{1}{2} t \sin t + \tfrac{1}{2}\sin t - \tfrac{1}{2} t \cos t.$$

$$= \cos t - \tfrac{1}{2}\sin t + \tfrac{1}{2} t \sin t - \tfrac{1}{2} t \cos t.$$

① $\frac{d}{ds} \frac{1}{s^2+1} = -\frac{2s}{(s^2+1)^2}$.

Not required

② $\frac{d}{ds} \frac{s}{s^2+1} = \frac{(s^2+1) - 2s \cdot s}{(s^2+1)^2} = \frac{-s^2+1}{(s^2+1)^2} = \frac{-(s^2+1)+2}{(s^2+1)^2}$.

$$= -\frac{1}{(s^2+1)} + \frac{2}{(s^2+1)^2}.$$

① $\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = (-1)t\, \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$.

② $\mathcal{L}^{-1}\left\{-\frac{1}{s^2+1} + \frac{2}{(s^2+1)^2}\right\} = (-1)t\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$.

① $\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = -t\sin t. \Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t.}$

② $-\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = -t\cos t.$

$$-\sin t + 2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = -t\cos t.$$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}(\sin t - t\cos t).}$$

Then. $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t + \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$

Ex. $y^{(4)} - y = 0.$ $y(0) = 1$ $y'(0) = 0$ $y''(0) = 1$ $y^{(3)}(0) = 0$

Apply the Laplace transform.

$$s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y^{(3)}(0) - Y = 0.$$

$$(s^4 - 1) Y - s^3 - s = 0.$$

$$Y(s) = \frac{s^3 + s}{s^4 - 1} = \frac{s\cancel{(s^2+1)}}{(s+1)(s-1)\cancel{(s^2+1)}} = \frac{a}{s+1} + \frac{b}{s-1}.$$

$$s = a(s-1) + b(s+1).$$

Let $s = 1$ $\quad 2b = 1 \quad b = \frac{1}{2}$

$s = -1 \quad -2a = -1 \quad a = \frac{1}{2}.$

$$y(t) = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}.$$

$$= \frac{1}{2} e^{-t} + \frac{1}{2} e^{t}.$$

Laplace transform of systems of differential equation.

Ex1. $y_1' = -3y_1 + 4y_2 + \sin t$ $y_1(0) = 0$

$y_2' = -2y_1 + 3y_2 + t.$ $y_2(0) = 1.$

Take the Laplace transform of both equations. Let $\mathcal{L}\{y_1\} = Y_1$

$\mathcal{L}\{y_2\} = Y_2.$

$$sY_1 - 0 = -3Y_1 + 4Y_2 + \frac{1}{1+s^2}.$$

$$sY_2 - 1 = -2Y_1 + 3Y_2 + \frac{1}{s^2}.$$

$$\Rightarrow (s+3)Y_1 - 4Y_2 = \frac{1}{1+s^2}. \quad ①$$

$$2Y_1 + (s-3)Y_2 = \frac{1}{s^2} + 1. \quad ②$$

Solve for Y_1 & Y_2.

From ①: $4Y_2 = (s+3)Y_1 - \frac{1}{1+s^2} \Rightarrow Y_2 = \frac{s+3}{4}Y_1 - \frac{1}{4(1+s^2)}$

Plug this to ②: $2Y_1 + \frac{(s+3)(s-3)}{4}Y_1 - \frac{s-3}{4(1+s^2)} = \frac{s^2+1}{s^2}$

$$\frac{8+s^2-9}{4}Y_1 = \frac{s^2+1}{s^2} + \frac{s-3}{4(1+s^2)}$$

$$\frac{s^2-1}{4}Y_1 = \frac{4(s^2+1)^2 + s^2(s-3)}{4s^2(1+s^2)} = \frac{4s^4+s^3+5s^2+4}{4s^2(1+s^2)}$$

$$Y_1 = \frac{4s^4+s^3+5s^2+4}{s^2(s^2+1)(s+1)(s-1)}.$$

$$Y_2 = \frac{(s+3)(4s^4+s^3+5s^2+4)}{4s^2(s^2+1)(s+1)(s-1)} - \frac{1}{4(1+s^2)}$$

$$Y_2 = \frac{(S+3)(4S^4+S^3+5S^2+4) - S^2(S+1)(S-1)}{4S^2(S^2+1)(S+1)(S-1)}$$

$$Y_2 = \frac{4S^5+12S^4+8S^3+16S^2+4S+12}{4S^2(S^2+1)(S+1)(S-1)}$$

$$Y_2 = \frac{S^5+3S^4+2S^3+4S^2+S+3}{S^2(S^2+1)(S+1)(S-1)}.$$

Compute $y_1 = \mathcal{L}^{-1}\{Y_1\}$ using partial fraction.

$$Y_1 = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S+1} + \frac{D}{S-1} + \frac{ES+F}{S^2+1}$$

$$4S^4+S^3+5S^2+4 = AS(S^2+1)(S+1)(S-1) + B(S^2+1)(S+1)(S-1)$$
$$+ CS^2(S^2+1)(S-1) + DS^2(S^2+1)(S+1)$$
$$+ (ES+F)S^2(S+1)(S-1)$$

Let $S=0$. $\quad -B=4 \quad B=-4.$

Let $S=1$ $\quad 4D=14 \quad D=\frac{7}{2}$

Let $S=-1$ $\quad -4C=12 \quad C=-3.$

$$\text{right} = (A+C+D+E)S^5 + (B-C+D+F)S^4$$
$$+ (C+D-E)S^3 + \cdots$$

$B-C+D+F=4 \quad -4+3+\frac{7}{2}+F=4 \quad \frac{5}{2}+F=4 \quad F=\frac{3}{2}$

$C+D-E=1 \quad \frac{7}{2}-3-E=1 \quad \frac{1}{2}-E=1 \quad E=-\frac{1}{2}$

$A+C+D+E=0.$ $\qquad A-3+\frac{7}{2}-\frac{1}{2}=0.$

$A=0.$

$$Y_1=\frac{-4}{s^2}-\frac{3}{s+1}+\frac{7}{2}\frac{1}{s-1}+\frac{-\frac{1}{2}s+\frac{3}{2}}{s^2+1}.$$

$$y_1(t)=-4t-3e^{-t}+\frac{7}{2}e^{t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t.$$

We can use partial fractions to compute. $y_2=\mathcal{L}^{-1}\{Y_2\}$

$$y_2(t)=\frac{7}{2}e^{t}-\frac{3}{2}e^{-t}+\sin t-3t-1$$

Ex. 2.

$$x''+y'+2x=0 \qquad x(0)=0 \quad x'(0)=0$$

$$2x'-y'=\cos t \qquad y(0)=0.$$

$$s^2X-\cancel{s\,x(0)}-\cancel{x'(0)}+sY-\cancel{y(0)}+2X=0$$

$$2(sX-\cancel{x(0)})-(sY-\cancel{y(0)})=\frac{s}{s^2+1}$$

$$(s^2+2)X+sY=0. \qquad ①$$

$$2s\,X-sY=\frac{s}{s^2+1} \qquad ②.$$

From ① $sY=-(s^2-2)X \Rightarrow Y=-\frac{s^2+2}{s}X$

Plug this to ②. $\qquad 2sX+(s^2+2)X=\frac{s}{s^2+1}$

$$(s^2+2s+2)X=\frac{s}{s^2+1}$$

$$X=\frac{s}{(s^2+1)(s^2+2s+2)}=\frac{s}{(s^2+1)[(s+1)^2+1]}$$

Compute $x = \mathcal{L}^{-1}\{X\}$.

Let $X = \dfrac{S}{(S^2+1)[(S+1)^2+1]} = \dfrac{AS+B}{S^2+1} + \dfrac{CS+D}{[(S+1)^2+1]}$

$$S = (AS+B)(S^2+2S+2) + (CS+D)(S^2+1)$$

$$S = AS^3+2AS^2+2AS + BS^2+2BS+2B+CS^3+CS+DS^2+D$$

$$S = (A+C)S^3 + (2A+B+D)S^2 + (2A+2B+C)S$$

$$+ 2B+D.$$

$A + C = 0$. $\quad C = -A$

$2A+B \quad +D = 0$. $\quad 2A+B-2B = 0 \quad 2A-B=0. \Rightarrow B=2A$

$2A+2B+C \quad = 1 \quad 2A+2B-A = 1 \quad A+2B = 1$

$2B \quad +D = 0$. $\quad D = -2B \quad$ Then $A+4A = 1$

$A = \frac{1}{5}$.

$B = \frac{2}{5} \quad C = -\frac{1}{5} \quad D = -\frac{4}{5}$

$$X = \frac{\frac{1}{5}S+\frac{2}{5}}{S^2+1} + \frac{-\frac{1}{5}S - \frac{4}{5}}{(S+1)^2+1} \Rightarrow \frac{-\frac{1}{5}(S+1)+\frac{1}{5}-\frac{4}{5}}{(S+1)^2+1}$$

$$= \frac{1}{5}\frac{S}{S^2+1} + \frac{2}{5}\frac{1}{S^2+1} - \frac{1}{5}\frac{S+1}{(S+1)^2+1} - \frac{3}{5}\frac{1}{(S+1)^2+1}$$

$$x(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t.$$

$$Y = -\frac{s^2+2}{s} X = \frac{-s^2-2}{(s^2+1)[(s+1)^2+1]}$$

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{(s+1)^2+1}$$

Then $-s^2-2 = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$

$$= (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + 2B+D.$$

$A+C = 0$ $\qquad C=-A$

$2A+B+D=-1$ $\qquad 2A+B-2B-2=-1$ $\qquad 2A-B=1$

$2A+2B+C=0.$ $\qquad 2A+2B-A=0$ $\qquad A+2B=0.$ $\qquad A=-2B.$

$2B+D=-2.$ $\qquad D=-2-2B$ $\qquad$ Then $-4B-B=1$

$B=-\frac{1}{5}$

$A=\frac{2}{5}$ $\quad C=-\frac{2}{5}$ $\quad D=-2-2(-\frac{1}{5}) = -2+\frac{2}{5} = -\frac{8}{5}$

$$Y = \frac{\frac{2}{5}s-\frac{1}{5}}{s^2+1} + \frac{-\frac{2}{5}s-\frac{8}{5}}{(s+1)^2+1} \Rightarrow \frac{-\frac{2}{5}(s+1)+\frac{2}{5}-\frac{8}{5}}{(s+1)^2+1}$$

$$Y = \frac{2}{5}\frac{s}{s^2+1} - \frac{1}{5}\frac{1}{s^2+1} - \frac{2}{5}\frac{s+1}{(s+1)^2+1} - \frac{6}{5}\frac{1}{(s+1)^2+1}$$

$$y(t) = \frac{2}{5}\cos t - \frac{1}{5}\sin t - \frac{2}{5}e^{-t}\cos t - \frac{6}{5}e^{-t}\sin t$$