Chapter 4. Second order linear equations

The general form of a 2nd order equation is

$$y'' = f(t, y, y')$$

In general, the solution has two constants.

Inireal value $\quad y(t_0) = y_0 \qquad y'(t_0) = y_1$

Linear equations

$$y'' + p(t)y' + q(t)y = g(t).$$

If $g(t) = 0$, the linear equation is homogeneous.

If $g(t) \neq 0$, ———————— non-homogeneous.

If $p(t)$, $q(t)$ are constants. we say the equation has constant coefficients.

otherwise, the equation has variable coefficients.

Transform a second order equation into first order system of equations.

Let $x_1 = y$

$\cancel{x_2} = y'$

then $\dfrac{dx_1}{dt} = y' = x_2$.

$$\frac{dx_2}{dt} = y'' = -q(t)y - p(t)y' + g(t) = -q(t)x_1 - p(t)x_2 + g(t).$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

Ex. Solve. $y'' + 7y' + 10y = 0$.

Let $x_1 = y \quad x_2 = y'$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find eigenvalue & eigenvector of A.

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -10 & -7-\lambda \end{bmatrix} \qquad \det(A-\lambda I) = -\lambda(-7-\lambda) - (-10) = 7\lambda + \lambda^2 + 10$$
$$= (\lambda+2)(\lambda+5)$$

$\lambda_1 = -2$. $\quad \begin{bmatrix} 2 & 1 \\ -10 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 = -5 \quad \begin{bmatrix} 5 & 1 \\ -10 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

General Sol. $y(t) = x_1(t) = c_1 e^{-2t} + c_2 e^{-5t}$

Existence and uniquess for. $y'' + p(t)y' + q(t)y = g(t)$.

with initial value $\quad y(t_0) = y_0 \quad y'(t_0) = y_1$.

Theorem 4.2.1. Let $p(t)$, $q(t)$ & $g(t)$. be continuous. on an open interval I containing t_0, Then ~~there~~ exits a unique solution $y = \phi(t)$. throughout the interval I.

Ex. Consider the IVP.

$$(\sin t)\, y'' + \frac{1}{t-1}y' - e^t y = 0$$

$y(2) = 3 \qquad y'(2) = 5$.

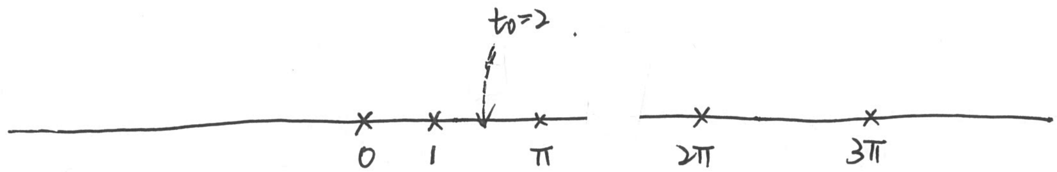
Find the longest interval in which. the solution to the IVP is certain to exist.

Write the equation in standard form.

$$y'' + \underbrace{\frac{1}{(t-1)(\sin t)}}_{p(t)} y' - \underbrace{\frac{e^t}{\sin t}}_{q(t)} y = 0.$$

$p(t)$ is continuous except $t = 1 \quad 0 \quad \pi \quad -\pi \quad 2\pi \quad -2\pi \quad \cdots\cdots$

$q(t)$ is continuous except $t = \quad 0 \quad \pi \quad -\pi \quad 2\pi \quad -2\pi \quad \cdots$



A Unique Solution exist on $(1, \pi)$.

Theory for homogeneous equations: $y'' + p(t)y' + q(t)y = 0.$

- Superposition principle.

Corollary 4.2.3. If $y_1(t)$ and $y_2(t)$ are two solutions to the homogeneous eq.

$$y'' + p(t)y' + q(t)y = 0$$

Then the linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1 and c_2.

Proof: If y_1 & y_2 are solutions. $\quad y_1'' + p(t)y_1 + q(t)y_1 = 0.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad y_2'' + p(t)y_2 + q(t)y_2 = 0.$

Then $(c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) =$

$$= c_1y_1'' + c_1p(t)y_1 + c_1q(t)y_1 + c_2y_2'' + c_2p(t)y_2' + c_2q(t)y_2$$

$$= c_1\underbrace{(y_1'' + p(t)y_1 + q(t)y)}_{=0} + c_2\underbrace{(y_2'' + p(t)y_2' + q(t)y_2)}_{=0}$$

$$= 0.$$

Recall that by letting $x_1 = y$
$x_2 = y'$

The homogeneous equation is equivalent to

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Two solutions $\vec{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}$ $\vec{x}_2(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$ are linearly independent.

if $W[\vec{x}_1\ \vec{x}_2](t) = \det \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} \neq 0.$

Let $y_1(t)$ & $y_2(t)$ be two solutions. $y'' + p(t)y' + q(t)y = 0$.

Def [Wronskian of y_1 & y_2]

$$W[y_1\ y_2](t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Def [Linear independence]

$y_1(t)$ & $y_2(t)$ are independent if $W[y_1\ y_2](t) \neq 0$.

General solution

Theorem 4.2.7. If $y_1(t)$ & $y_2(t)$ are two independent solutions of

$$y'' + p(t)y' + q(t)y = 0.$$

Then y_1 & y_2 form a fundamental set of solutions, and the general solution of this homogeneous equation is given by $y = C_1 y_1(t) + C_2 y_2(t)$.

Initial Value Problem. If there are given initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$, then these conditions determine C_1 & C_2 uniquely.

Abel's Theorem Theorem 4.2.8.

Part (a) The Wronskian of two solutions of the system

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}$$

is given by.

$$W(t) = c\,e^{\int tr(P)(t)\,dt} = c\,e^{\int (P_{11}(t) + P_{22}(t))\,dt}.$$

where c is a constant that depends on the pair of solutions.

Part (b) The Wronskian of two solutions of the second order equation.

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

is given by

$$W(t) = c\,e^{-\int p(t)\,dt}.$$

where c is a constant that depends on the pair of solutions.

How to derive Part (b) from Part (a) ?

$$P(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}$$

Ex. Find the Wronskian of any pair of solutions of.

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{P(t)} \vec{x}$$

$$W(t) = c\,e^{\int tr(P)(t)\,dt} = c\,e^{\int 2\,dt} = c\,e^{2t}.$$

This c depends on the pair of solutions.

For $\vec{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \vec{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. $\quad W(t) = e^{2t} \det\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = -4e^{2t}$

Ex. Find the Wronskian of any pair of solutions of

$$(1-t)y'' + ty' - y = 0$$

First, Write this eqation in standar form.

$$y'' + \underbrace{\frac{t}{1-t}}_{p(t)} y' - \underbrace{\frac{1}{1-t}}_{q(t)} y = 0$$

$$-\int p(t)\,dt = -\int \frac{t}{1-t}\,dt = \int \frac{1-u}{u}\,du = \ln|u| - u$$

let $u = 1-t$ $\quad du = -dt$

$$= t-1+\ln|1-t|$$

$$W[y_1\ y_2](t) = Ce^{-\int p(t)dt} = Ce^{-t}e^{1}e^{\ln|1-t|}$$

$$= Ce|1-t|e^{-t} = \tilde{C}(t-1)e^{-t}$$

For $y_1(t) = t$ $\quad y_2(t) = e^t$

$$W[y_1\ y_2](t) = \begin{bmatrix} t & e^t \\ 1 & e^t \end{bmatrix} = (t-1)e^t$$

Corollary 4.2.9. The Wronskian is either never zero or always zero.

Section 4.3. Linear homogeneous equations with constant coefficients.

?? How to solve. $ay'' + by' + cy = 0$.

This equation is equivalent to

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}}_{A} \vec{x} \quad \text{by letting} \quad \begin{array}{l} x_1 = y \\ x_2 = y' \end{array}$$

Find eigenvalues & eigenvectors of A

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda\left(-\frac{b}{a} - \lambda\right) + \frac{c}{a} = \frac{b}{a}\lambda + \lambda^2 + \frac{c}{a}$$

$$= \frac{1}{a}\left(a\lambda^2 + b\lambda + c\right)$$

Two eigenvalues λ_1, λ_2 are roots of $a\lambda^2 + b\lambda + c = 0$.

Eigenvectors. $\begin{bmatrix} -\lambda & 1 \\ * & * \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$

Solution $\vec{x}(t) = e^{\lambda t}\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$ where λ is a root of $a\lambda^2 + b\lambda + c = 0$

$$y(t) = x_1 = e^{\lambda t}.$$

Thm. 4.3.1. If λ is a root of $\boxed{a\lambda^2 + b\lambda + c = 0}$ (Characteristic equation of $ay'' + by' + c = 0$), then $y(t) = e^{\lambda t}$ is a solution to

$$ay'' + by' + c = 0.$$

Roots of $a\lambda^2 + b\lambda + c = 0$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Case 1. $b^2 - 4ac > 0$, the roots are real & distinct $\lambda_1 \neq \lambda_2$.

Case 2. $b^2 - 4ac = 0$, the roots are real & equal $\lambda_1 = \lambda_2$.

Case 3. $b^2 - 4ac < 0$, the roots are complex conjugate.

$$\lambda_1 = \mu + \nu i \qquad \lambda_2 = \mu - \nu i$$

Case 1. $y_1(t) = e^{\lambda_1 t}$ $\quad y_2(t) = e^{\lambda_2 t}$.

Distinct real roots. $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.

Case 2.

Repeated roots. $y_1(t) = e^{\lambda t}$ $\quad y_2(t) = t e^{\lambda t}$ Page. 230.

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

Case 3. $\lambda_1 = \mu + \nu i$ $\quad \lambda_2 = \mu - \nu i$

Complex roots. $z_1(t) = e^{(\mu + \nu i)t}$ $\quad z_2(t) = e^{(\mu - \nu i)t}$.

$$z_1(t) = e^{\mu t}(\cos \nu t + i \sin \nu t)$$

$$y_1(t) = e^{\mu t} \cos \nu t \qquad y_2(t) = e^{\mu t} \cos \nu t.$$

$$y(t) = c_1 e^{\mu t} \cos \nu t + c_2 e^{\mu t} \cos \nu t.$$

Ex. Find the general solution of.

a) $y'' + 5y' + 6y = 0$ b) $y'' - y' + \frac{1}{4}y = 0$

c) $y'' + y' + y = 0$ d) $y'' + 9y = 0$

a) Charateristic equation. $\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+2)(\lambda+3) = 0$ $\lambda_1 = -2$ $\lambda_2 = -3$

$y(t) = C_1 e^{-2t} + C_2 e^{-3t}$.

b). Characteristic equation $\lambda^2 - \lambda + \frac{1}{4} = 0$.

$(\lambda - \frac{1}{2})^2 = 0$ $\lambda_1 = \lambda_2 = \frac{1}{2}$.

$y(t) = C_1 e^{\frac{1}{2}t} + C_2 t e^{\frac{1}{2}t}$

c) Characteristic equation. $\lambda^2 + \lambda + 1 = 0$.

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$ Complex.

$\mu = -\frac{1}{2}$. $\nu = \frac{\sqrt{3}}{2}$

$$y(t) = C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

d) Characteristic equation. $\lambda^2 + 9 = 0$.

$\lambda_1 = 3i$ $\lambda_2 = -3i$ complex

$\mu = 0$ $\nu = 3$.

$y(t) = C_1 \cos 3t + C_2 \sin 3t$.

Section 4.5. Nonhomogeneous equations

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

Theorem 4.5.1. If $Y_1(t)$ & $Y_2(t)$ are two solutions of $y'' + p(t)y' + q(t)y = g(t)$, then $Y_1 - Y_2$ is a solution to the homogenous equation $y'' + p(t)y' + q(t)y = 0$. If $y_1(t)$ & $y_2(t)$ form a fundamental set of solutions of the homogenous equation, then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t).$$

Proof.

$$Y_1'' + p(t)Y_1' + q(t)Y_1 = g(t)$$

$$Y_2'' + p(t)Y_2' + q(t)Y_2 = g(t)$$

Substract two equation and obtain.

$$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) = 0.$$

Theorem 4.5.2. The general solution of the nonhomogeous equation is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t).$$

where $y_1(t)$ & $y_2(t)$ form a fundamental set of solutions of the homogeneous equation, and $Y(t)$ is a specific solution of the nonhomogeneous equation.

How to find a particular solution $Y(t)$?

☆ Method of underdetermined coefficients.

Table 4.5.1. The particular solution of $ay'' + by' + cy = g(t)$

$g(t)$	$Y(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)$
$P_n(t) e^{\alpha t}$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{\alpha t}$
$P_n(t) e^{\alpha t} \begin{cases} \sin\beta t \\ \cos\beta t \end{cases}$	$t^s \left[\begin{array}{l} (A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{\alpha t} \cos\beta t \\ + (B_0 t^n + B_1 t^{n-1} + \cdots + B_n) e^{\alpha t} \sin\beta t \end{array} \right]$

Here $s = 0, 1, 2$. the number of times. $\begin{cases} 0 \\ \alpha \\ \alpha + i\beta \end{cases}$ is a root of charateristic.

Ex 1. $y'' - 3y' - 4y = 3e^{2t}$.

Let $Y(t) = Ae^{2t}$

then $Y'(t) = 2Ae^{2t}$. $\quad Y''(t) = 4Ae^{2t}$.

$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$.

$-6Ae^{2t} = 3e^{2t} \qquad A = -\frac{1}{2}$.

So $Y(t) = -\frac{1}{2}e^{2t}$

homogeneous

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$.

$\lambda_1 = 4 \quad \lambda_2 = -1$

General $y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}$.

Ex 2. $y'' - 3y' - 4y = 2\sin t$.

Let $Y(t) = A\sin t + B\cos t$.

$Y'(t) = A\cos t - B\sin t$

$Y''(t) = -A\sin t - B\cos t$.

$-A\sin t - B\cos t - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$.

$(-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t$.

$(-5A + 3B)\sin t + (-3A - 5B)\cos t = 2\sin t$.

$\begin{cases} -5A + 3B = 2. \\ -3A - 5B = 0 \end{cases} \Leftrightarrow \begin{cases} -25A + 15B = 10. \\ -9A - 15B = 0. \end{cases} \qquad -34A = 10.$

$A = -\frac{5}{17} \qquad B = -\frac{3}{5}A = \frac{3}{17}$

$Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$.

General solution $y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t$.

Ex3. $y'' - 3y' - 4y = 4t^2 - 1$

Let $Y(t) = At^2 + Bt + C$.

$Y'(t) = 2At + B$

$Y''(t) = 2A$

$2A - 3(2At + B) - 4(At^2 + Bt + C) = 4t^2 - 1$

$-4At^2 + (-6A - 4B)t + (2A - 3B - 4C) = 4t^2 - 1$

$$\begin{cases} -4A = 4 \\ -6A - 4B = 0 \\ 2A - 3B - 4C = -1 \end{cases}$$

$A = -1$

$B = -\frac{6}{4}A = \frac{3}{2}$

$-2 - \frac{9}{2} - 4C = -1$

$4C = -1 - \frac{9}{2} = -\frac{11}{2}$

$C = -\frac{11}{8}$

$Y(t) = -\frac{11}{8}t^2 + \frac{3}{2}t - 1$

General sol. $y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{11}{8}t^2 + \frac{3}{2}t - 1$

Ex4. $y'' - 3y' - 4y = -8e^t \cos 2t$.

Let $Y(t) = Ae^t \cos 2t + Be^t \sin 2t$.

$Y'(t) = Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t$

$= (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t$.

$Y''(t) = (A + 2B)e^t \cos 2t - 2(A + 2B)e^t \sin 2t$

$+ (-2A + B)e^t \sin 2t + 2(-2A + B)e^t \cos 2t$

$= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t$

$$\begin{cases} -3A + 4B = -8 \\ -4A - 3B = 0 \end{cases} \Leftrightarrow \begin{matrix} -9A + 12B = -24 \\ -16A - 12B = 0 \end{matrix}$$

$-25A = -24$

$A = \frac{24}{25}$ $\quad B = -\frac{16}{12}A = -\frac{4}{3}A = -\frac{4}{3} \cdot \frac{24}{25} = -\frac{32}{25}$

$Y(t) = \frac{24}{25}e^t \cos 2t - \frac{32}{25}e^t \sin 2t$.

Ex5. $y'' - 3y' - 4y = 2e^{-t}$.

Solutions of homogeneous eq. $y_1(t) = e^{-t}$ $y_2(t) = e^{4t}$.

If let $Y(t) = Ae^{-t}$.

$Y'' - 3Y' - 4Y = 0$ no matter what A is.

Then let $Y(t) = Ate^{-t}$ $\boxed{s=1 \text{ in Table 4.5.1}}$

$Y'(t) = Ae^{-t} - Ate^{-t}$.

$Y''(t) = -Ae^{-t} - A(e^{-t} - te^{-t})$

$= -2Ae^{-t} + Ate^{-t}$.

$Y'' - 3Y' - 4Y$

$= -2Ae^{-t} + Ate^{-t} - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t}$

$= (-2A - 3A \quad)e^{-t} + (A + 3A - 4A)te^{-t}$

$= -5Ae^{-t}$

$-5A = 2$ $A = -\frac{2}{5}$ $Y(t) = -\frac{2}{5}te^{-t}$.

General sol. $y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{2}{5}te^{-t}$

Ex 6. $y'' = 3t^3 - t$.

Characteristic $\lambda^2 = 0$ $\lambda_1 = 0$ $\lambda_2 = 0$. $s = 2$.

$Y(t) = t^2(A_3t^3 + A_2t^2 + A_1t + A_0)$

Superposition principal.

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t.$$

particular solution.

Three equation $y'' - 3y' - 4y = 3e^{2t}$ $Y_1(t)$

$y'' - 3y' - 4y = 2\sin t$. $Y_2(t)$

$y'' - 3y' - 4y = -8e^t \cos 2t$ $Y_3(t)$

$Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$

Ex 7. $y'' - y' - 2y = -3te^{-t} + 2\cos 4t$.

Charateristic. $\lambda^2 - \lambda - 2 = 0$

$(\lambda-2)(\lambda+1)=0 \quad \lambda_1 = 2 \quad \lambda_2 = -1$

$g_1(t) = -3te^{-t} \quad s=1 \quad Y_1(t) = t(A_1 t + A_0)e^{-t}$

$g_2(t) = 2\cos 4t \quad s=0 \quad Y_2(t) = B\cos 4t + C\sin 4t$

Ex 8. $y'' + 2y' + 5y = t^2 e^{-t} \sin 2t$.

Charatteristic $\lambda^2 + 2\lambda + 5 = 0$.

$$\lambda = \frac{-2 \pm \sqrt{4-4\cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$\alpha = -1 \quad \beta = 2 \quad e^{-t}\cos 2t. \quad e^{-t}\sin 2t.$

$s=1 \quad Y(t) = t\left[(A_2 t^2 + A_1 t + A_0) e^{-t} \cos 2t + (B_2 t^2 + B_1 t + B_0) e^{-t} \sin 2t. \right]$

Ex 9. $y'' + y = \tan t$.

Method of underdetermined coefficients does not apply.