Chapter 2. 1st order ODE.

The general form of 1st order. ODE is

$$\frac{dy}{dx} = f(x, y)$$

Def [Separable 1st order ODE]

If the right hand side $f(x,y)$ can be written as $p(x)\, q(y)$, then the I
is called separable.

eg: $y' = \frac{x^2 + \sin x}{1+y^2} = (x^2 + \sin x)\left(\frac{1}{1+y^2}\right)$ separable.

$y' = xy + 1$, is not separable.

$y' = ry - k = r\left(y - \frac{k}{r}\right)$. separable.

Solve. separable 1st order ODE.

$$\frac{dy}{dx} = p(x)\, q(y)$$

$$\int \frac{dy}{q(y)} = \int p(x)\, dx$$

Eg: $y' = \frac{x^2 + \sin x}{1+y^2}$ $y(0) = 1$ Solve this IVP.

$$\frac{dy}{dx} = \frac{x^2 + \sin x}{1+y^2}$$

$$\int (1+y^2)\, dy = \int (x^2 + \sin x)\, dx$$

$$y + \frac{1}{3}y^3 = \frac{1}{3}x^3 - \cos x + C.$$

So the general solution is. $y + \frac{1}{3}y^3 = \frac{1}{3}x^3 - \cos x + C$. which is an implicit functi of y.

Initial value: $x=0$ $y=1$ plugin.

$1 + \frac{1}{3} = -\cos 0 + C$ $\quad \frac{4}{3} = -1 + C$ $\quad C = \frac{7}{3}$

So the solution to IVP is. $y + \frac{1}{3}y^3 = \frac{1}{3}x^3 - \cos x + \frac{7}{3}$

Why does this work?

$$\int \frac{1}{q(y(x))} y'(x)\, dx = \int p(x)\, dx + C. \qquad dy = y'(x)\, dx$$

$$\Rightarrow \int \frac{1}{q(y)}\, dy = \int p(x)\, dx + C.$$

$\boxed{\text{1 st order linear DE}}$

Recall: the general form of a linear DE of order n. is

$$a_n(t)\, y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0(t) y = g(t).$$

So a 1st order. DE can be written as.

$$y'(t) + p(t)\, y = g(t).$$

Examples: In the mice-owl example.

$$p'(t) = rp - k.$$

$$p'(t) \boxed{-r} p = \boxed{-k}$$

Newton's law of law of colling. : $u(t)$ temperature of an object.

$$u'(t) = -k(u - T_0).$$

$$u'(t) \boxed{+k} u = \boxed{kT_0}$$

How to solve 1st order. linear ODE?

Note: 1st order linear ODE may not be separable.

Ex: $\frac{dy}{dt} + 5y = e^t$ not separable.

Def [Integrating factor for 1st order linear ODE]

$$\mu(t) = e^{\int p(t)\, dt}$$

or $\frac{d\mu(t)}{dt} = p(t)\, \mu \Rightarrow \int \frac{d\mu}{\mu} = \int p(t)\, dt$

$$\ln|\mu| = \int p(t)\, dt$$

$$\mu = C e^{\int p(t)\, dt}.$$

HOW to solve. 1st order linear ODE?

1. Put the equation in standard form $y' + p(t)y = g(t)$.

2. Calculate the integrating factor. $\mu(t) = e^{\int p(t)dt}$

3. Multiply both sides by $\mu(t)$

$$\therefore \mu(t)\frac{dy}{dt} + \underbrace{\boxed{p(t)\mu(t)}}_{\frac{d\mu}{dt}} y = \mu(t)g(t).$$

$$\int \frac{d}{dt}\left(\mu(t)y(t)\right) = \int \mu(t)g(t).$$

$$\mu(t)y(t) = e^{\int \mu(t)g(t)dt} + C.$$

4. Solution $\quad y(t) = \frac{1}{\mu(t)} e^{\int \mu(t)g(t)dt} + \frac{C}{\mu(t)}.$

Ex: Solve. $\frac{dy}{dt} + \overset{p(t)}{\widetilde{5}}y = \overset{g(t)}{\widetilde{e^{t}}}.$

1. Standar form ✓

2. Integrating factor. $\mu(t) = e^{\int p(t)dt} = e^{\int 5dt} = e^{5t}$

3. $$\underbrace{e^{5t}\frac{dy}{dt} + 5e^{5t}y}_{} = e^{t}e^{5t} = e^{6t}.$$

$$\int \frac{d}{dt}\left(e^{5t}y\right) = \int e^{6t}$$

$$e^{5t}y(t) = \frac{1}{6}e^{6t} + C.$$

$$y(t) = \frac{1}{6}e^{t} + Ce^{-5t}.$$

Ex: Solve the IVP. $ty' + (t+1)y = 2e^{-2t}$ $y(1)=0$.
and determine the interval in which the solution exists.

Step 1: Standard form.

$$y' + \frac{t+1}{t} y = \frac{2e^{-2t}}{t}$$

Step 2. Integrating factor.

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{t+1}{t} dt} = e^{\int (1+\frac{1}{t}) dt} = e^{t+\ln|t|}$$
$$= |t|e^{t}.$$

Step 3. Multiply both sides by $\mu(t)$

$$|t|e^t y' + \frac{t+1}{t}|t|e^t y = \frac{2e^{-2t}}{t}|t|e^t$$

$$te^t y' + (t+1)e^t y = 2e^{-2t}e^t.$$

$$\int \frac{d}{dt}(te^t y) = \int 2e^{-t}$$

$$te^t y = -2e^{-t} + c$$

$$y(t) = \frac{-2e^{-t}+c}{te^t} = -\frac{2}{t}e^{-2t} + \frac{c}{te^t}$$

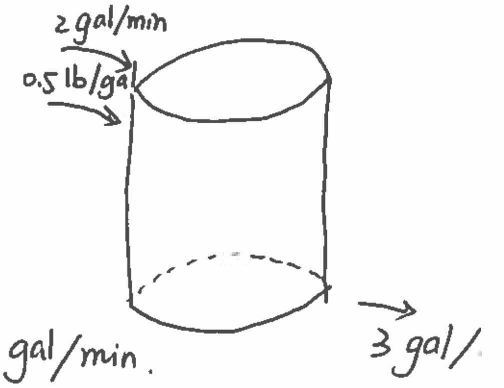
Step 4. IVP. $y(1) = -2e^{-2} + \frac{c}{e} = 0 \Rightarrow c = 2e^{-1}$

Solution to IVP. $y(t) = -\frac{2}{t}e^{-2t} + \frac{2e^{-1}}{te^t}$

Solution exists for $t > 0$

2.3 Modeling with 1 st order Linear ODE.

- The drink dispenser initially contains 1 gallon unsweet tea.
- Sugar water is added at a rate 2 gal/min sugar concentration. 0.5 lb/gal.
- Someone is using the dispenser at rate 3 gal/min.



Q: How much sugar is there in the tea after 20 sec?

Indepent variable: t (sec)

Dependent variable: Q (lb).

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}.$$

rate of change for sugar — rate in: $2\,gal/min \times 0.5\,lb/gal$ $\left(\frac{gal}{min} \cdot \frac{lb}{gal}\right)$ = 1 lb/min. — rate out: $3\,gal/min. \times$ sugar concen $\left(\frac{gal}{min} \times \frac{lb}{gal}\right)$

At time t, sugar concentration $= \dfrac{\text{amount sugar}}{\text{volumn of tea}} = \dfrac{Q}{1+2t-3t}$

$$\frac{dQ}{dt} = 1 - \frac{3Q}{1-t}$$

1 st order linear ODE

$Q(0) = 0$.

How to solve it?

Step 1: standard form.

$$\frac{dQ}{dt} + \underbrace{\frac{3}{1-t}}_{p(t)} Q = \underbrace{1}_{g(t)}$$

Step 2: Integrating factor.

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$u = 1-t$

$du = -dt$

$-\int \frac{3}{u}\, du$

$= -3\ln u.$

$$\mu(t) = e^{\int p(t)\, dt} = e^{\int \frac{3}{1-t}\, dt} = e^{-3\ln(1-t)}$$

$$= (1-t)^{-3}$$

Step 3. Multiplying both sides by the integrating factor.

$$(1-t)^{-3} \frac{dQ}{dt} + \frac{3}{1-t}(1-t)^{-3} Q = (1-t)^{-3}$$

$$\int \frac{d}{dt}\left((1-t)^{-3} Q\right) = \int (1-t)^{-3}\, dt = \int \frac{1}{(1-t)^3}\, dt.$$

$u = 1-t$

$du = -dt$

$-\int \frac{1}{u^3}\, du = -\int u$

$= -\frac{1}{-2} u^{-2}$

$$(1-t)^{-3} Q = -\frac{1}{2}(1-t)^{-2} + C.$$

$$Q = -\frac{1}{2}(1-t) + C(1-t)^3$$

Initial value: $Q(0) = -\frac{1}{2} + C = 0 \qquad C = \frac{1}{2}.$

$$Q(t) = -\frac{1}{2}(1-t) + \frac{1}{2}(1-t)^3.$$

Car loan: $S(t)$ balance due on the loan at any time t. Unit dollars.

t time unit year.

- accumulation of interest increases $S(t)$ — r annual interest rate.
- the payments by the borrower reduces $S(t)$. — k monthly payment rate.

$$\frac{dS}{dt} = rS - 12k.$$

Section 2.4. Linear and nonlinear differential equations.

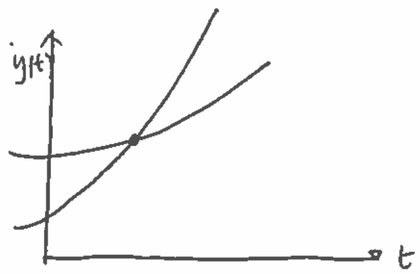
Today: existence and uniqueness of sol'ns. of 1 st order DE.

For the IVP. $\frac{dy}{dt} = f(t,y)$. with initial condition $y(t_0) = y_0$.

We want to know:

- does a solution exist? in which interval?
- Is the solution unique?

Qestion.

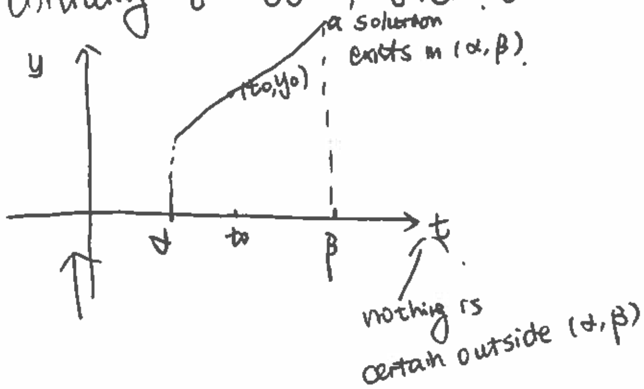


Can there be two integral curves passing through the same point?

For linear 1st order ODE, $\begin{cases} \frac{dy}{dt} + p(t) y = g(t) \\ p(t_0) = y \end{cases}$, we have the following.

Theorem 2.4.1

If the functions $p(t)$ and $g(t)$ are continuous on an open interval $I = (\alpha, \beta)$ containing $t = t_0$, then there exists a unique solution to the IVP on



Reason: If $p(t)$ is continuous, the integrating factor $\mu(t) = e^{\int_{t_0}^{t} p(t)\,dt}$ must be continuous, finite, >0.

So the general solution is

$$y = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(s) g(s) \, ds + C \right)$$

To make $y(t_0) = y_0$, we only have one choice of C, so the sol. is unique.

Ex: For the IVP. $ty' + e^t y = t^2 \tan t$. $y(1) = 5$.

Use the theorem to determine an interval in which the sol. of the given IVP. is certain to exist.

Step 1. Standard-form.

$$\frac{dy}{dt} + \underbrace{\frac{e^t}{t}}_{p(t)} y = \underbrace{t \tan t}_{g(t)}.$$

Step 2. Look at the continuity of $p(t)$ and $g(t)$

$p(t)$ is continuous in $(-\infty, 0) \cup (0, +\infty)$

$g(t)$ is continuous in. $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{3}{2}\pi)$, ...

except $\frac{\pi}{2}k$ k odd.

So. $p(t)$ & $g(t)$ are both continuous in $(0, \frac{\pi}{2})$.

By Theorem 2.4.1. A unique solution exists in $(0, \frac{\pi}{2})$

?? What if $y(2) = 5$: a unique solution exicts in $(\frac{\pi}{2}, \frac{3}{2}\pi)$.

<u>Nonlinear DE</u>

For nonlinear DE, we have the following.

Theorem 2.4.2. For the IVP. $y' = f(t, y)$

$y(t_0) = y_0$.

If $f(t,y)$ and $\frac{\partial f(t,y)}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$. $\gamma < y <$, then the IVP has a unique solution in some interval containing (t_0, y_0), then the IVP has a unique solution in some interval. $(t_0 - h, t_0 + h)$ contained in (α, β).

$$\frac{dy}{dt} = t^2 + t y^3 \leftarrow f(t,y) \qquad \frac{\partial f}{\partial y} = 3ty^2$$

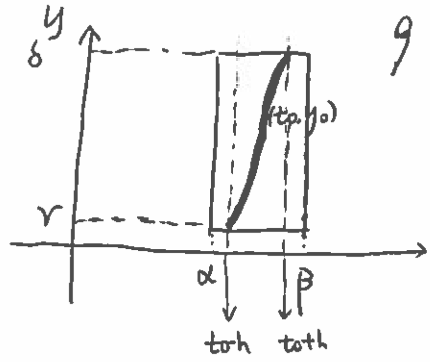
Note ① $\frac{\partial f}{\partial y}$ is the partial derivative of f w.r.t y.

where t is treat as constant.

② the theorem does not tell us what is h.

it may be small. Also nothing is certain.

outside (t_0-h, t_0+h).



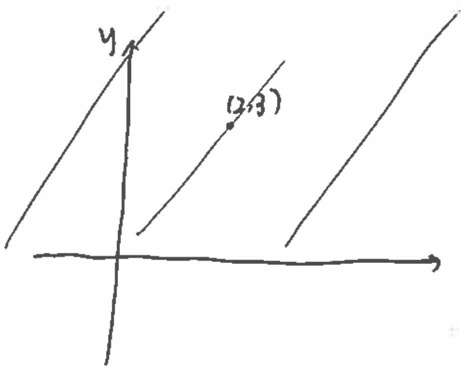
Ex: IVP. $\frac{dy}{dt} = y^{1/3}$ $\quad y(2) = 3$.

Does there exist a unique sol to this IVP in some small interva

containing $t_0 = 2$?

$f(y) = y^{1/3}$ continuous. $-\infty < t < +\infty$ $\quad -\infty < y < +\infty$.

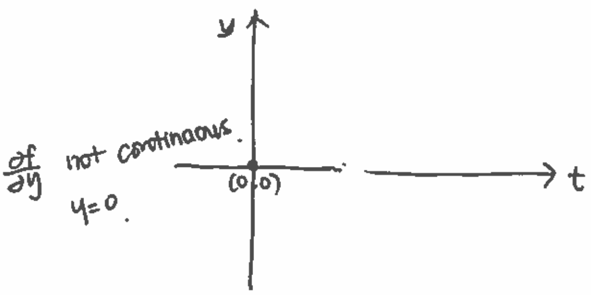
$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}}$ Continous $-\infty < t < +\infty$. $\quad y < 0$ and $y > 0$.



Box
$-\infty < t < +\infty$.
$0 < y < +\infty$.

Yes, a unique solution exist in $(2-h, 2+h)$ for some $h > 0$.

What if $y(0) = 0$



Can't draw a box containing $(2,0)$ where both $f(t,y)$ & $\frac{\partial f(t,y)}{\partial y}$ are continuous.

Indeed. the solution for $\begin{cases} y' = y^{1/3} \\ y(0)=0 \end{cases}$ is not unique.

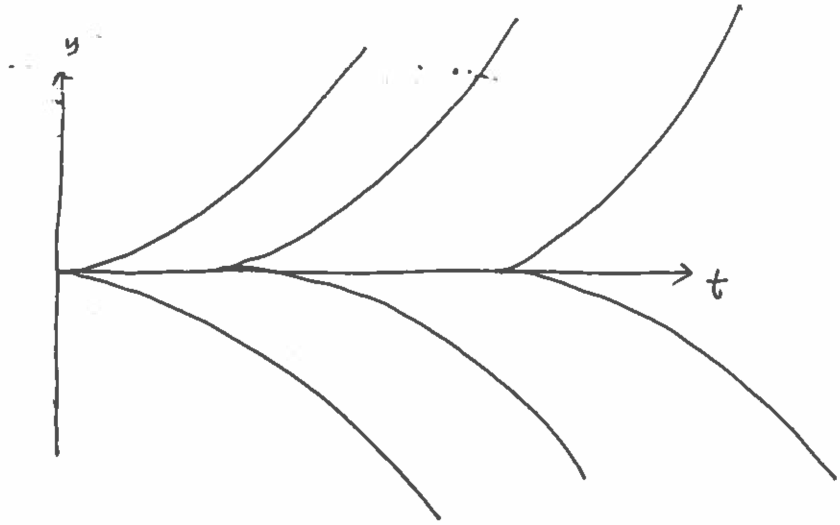
Ex: Solve $\begin{cases} \frac{dy}{dt} = y^{1/3} \\ y(0) = 0. \end{cases}$ Separable DE.

$$\int y^{-\frac{1}{3}}\, dy = \int dt.$$ $\boxed{\text{Separation of variables.}}$

$$\frac{3}{2} y^{\frac{2}{3}} = t + C.$$

$$y = \pm\left[\frac{2}{3}(t+C)\right]^{\frac{3}{2}}$$

$y(0)=0.$ $y = \left(\frac{2}{3}C\right)^{\frac{3}{2}} = 0.$ $C=0.$ $y = \pm\left(\frac{2}{3}t\right)^{\frac{3}{2}}$ $t \geq 0.$



- $y=0$ is also a solution.

- $y = \begin{cases} 0 & 0 \leq t < T. \\ \pm\left[\frac{2}{3}(t-T)\right]^{\frac{3}{2}}. & t \geq T. \end{cases}$ is also a solution

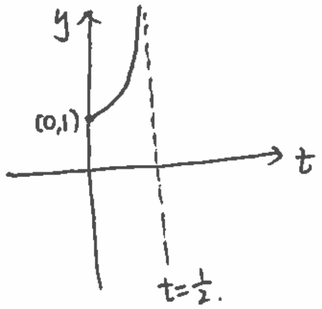
Also, even if $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous everywhere, the solution may stop existing somewhere.

Ex: $y' = y^2$ with $y(0) = 1$.

$\int \frac{dy}{y^2} = \int 1 \, dt$ $\quad -\frac{1}{2} y^{-1} = t + C$ $\quad -\frac{1}{2y} = t + C$.

$t=0 \quad y=1$ $\quad -\frac{1}{2} = C$. $\quad -\frac{1}{2y} = t - \frac{1}{2}$ $\quad -2y = \frac{1}{t-\frac{1}{2}}$ $\quad y = -\frac{1}{2t-1}$

$y = \frac{1}{1-2t}$. $\quad \lim_{t \to \frac{1}{2}^-} y(t) = +\infty$.



Solution exists in $(-\infty, 1)$.

Section 2.5 Autonomous DE and models for population dynamics.

Autonomous DE: $$\frac{dy}{dt} = f(y).$$

Exponential growth of population.

The rate of charge is proportional to the current population with growth rate r.

$$\frac{dy}{dt} = ry \qquad \frac{dy}{y} = r\,dt \qquad \ln|y| = rt \qquad y = Ce^{rt}.$$

In reality, exp growth can't continue forever due to lack of food or resour

$$\frac{dy}{dt} = h(y)\,y.$$ What kind of $h(y)$ gives the right model?

$h(y)$ should decreases as y increases.

Logistic eq. $$\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right) y. \qquad r, k > 0$$

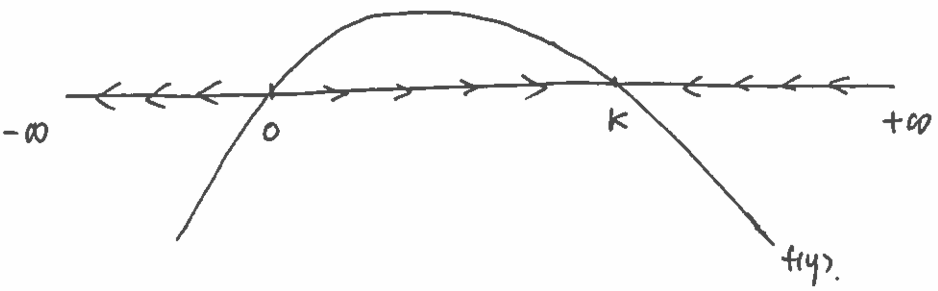
① Draw phase lines and sketch integral curves.

② Solve the IVP. with $y(0) = y_0$

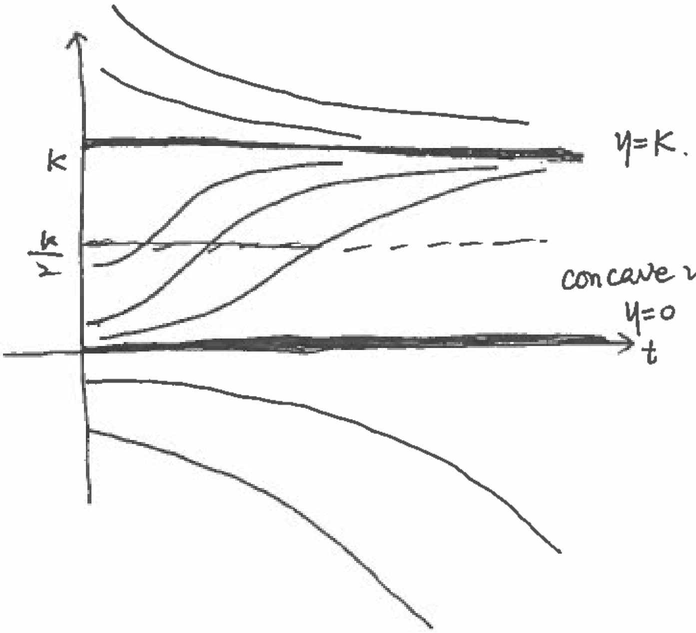
Equilibrium solutions. $r\left(1 - \frac{y}{k}\right) y = 0.$

$\boxed{y=0}$ unstable $\qquad \boxed{y=k}$ stable.

Phase lines.



Integral curves



Notice: Given IV $y(0) = y_0 > 0$ $\quad \lim_{t\to\infty} y(t) = K$.

So K is called environmental carrying capacity.

Solution's concavity.

$$y''(t) = \frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d}{dt} f(y(t)) = \frac{df}{dy}\,\frac{dy}{dt}.$$

$$f(y) = r\left(1-\frac{y}{K}\right)y = ry - \frac{r}{K}y^2 \qquad \frac{df}{dy} = r - \frac{2r}{K}y = r\left(1-\frac{2}{K}y\right)$$

$$\frac{dy}{dt} = f(y) = r\left(1-\frac{y}{K}\right)y.$$

$$y''(t) = r^2\left(1-\frac{2}{K}y\right)\left(1-\frac{y}{K}\right)y.$$

When $0 < y < \frac{K}{2}$.	$\left(1-\frac{2}{K}y\right)\left(1-\frac{y}{K}\right)y$		$y''(t) > 0$	concave up
	$+ \quad + \quad +$			
When $\frac{K}{2} < y < K$	$- \quad + \quad +$		$y''(t) < 0$	concave dow
When $y > K$.	$- \quad - \quad +$		$y''(t) > 0$	concave up

Solve the logistic model DE.

$$\frac{dy}{dt} = r\left(1-\frac{y}{K}\right)y. \qquad \text{separable.}$$

$$\int \frac{dy}{\left(1-\frac{y}{K}\right)y} = \int r\,dt.$$

Partial fraction $\dfrac{1}{\left(1-\frac{y}{K}\right)y} = \dfrac{A}{1-\frac{y}{K}} + \dfrac{B}{y}$ determine A&B.

$$= \frac{Ay + B\left(1-\frac{y}{K}\right)}{\left(1-\frac{y}{K}\right)y} = \frac{B+\left(A-\frac{B}{K}\right)y}{\left(1-\frac{y}{K}\right)y}$$

Compare both sides. $B=1$ $\quad A-\frac{B}{K}=0$ $\quad A=\frac{1}{K}.$

$$\frac{1}{\left(1-\frac{y}{K}\right)y} = \frac{1/K}{\left(1-\frac{y}{K}\right)} + \frac{1}{y} =$$

$$\int \frac{1}{K-y}\,dy + \int \frac{1}{y}\,dy = \ln|y| - \ln\left|1-\frac{y}{K}\right|$$

$$\Rightarrow \ln\left|\frac{y}{1-\frac{y}{K}}\right| = rt + C. \qquad \frac{y}{1-\frac{y}{K}} = Ce^{rt}.$$

IVP: $y(0) = y_0.$ $\quad t=0 \ \ y=y_0$ $\quad \dfrac{y_0}{1-\frac{y_0}{K}} = C$

Explicit sol.

$$\frac{y}{1-\frac{y}{K}} = \frac{y_0}{1-\frac{y_0}{K}}e^{rt} \qquad \frac{K-y_0}{K}y = y_0\frac{K-y}{K}e^{rt}.$$

$$(K-y_0)y = y_0(K-y)e^{rt} \qquad (K-y_0)y + y_0e^{rt}y = Ky_0e^{rt}$$

$$y = \frac{Ky_0e^{rt}}{K-y_0+y_0e^{rt}} = \frac{Ky_0}{y_0+(K-y_0)e^{-rt}}.$$

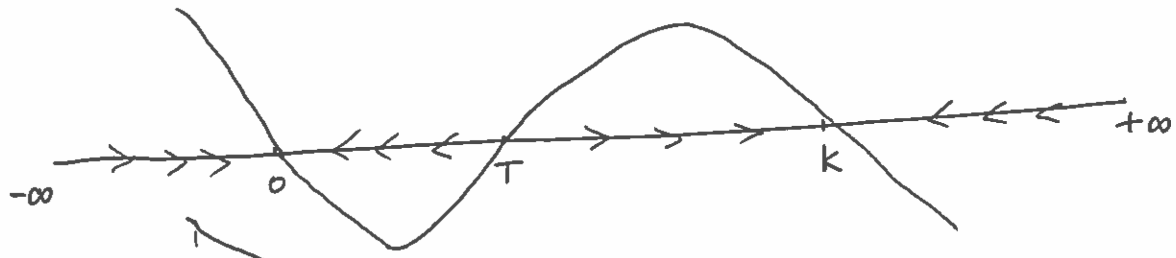
$$\lim_{t\to\infty} y(t) = K.$$

Logistic growth with a threshold.

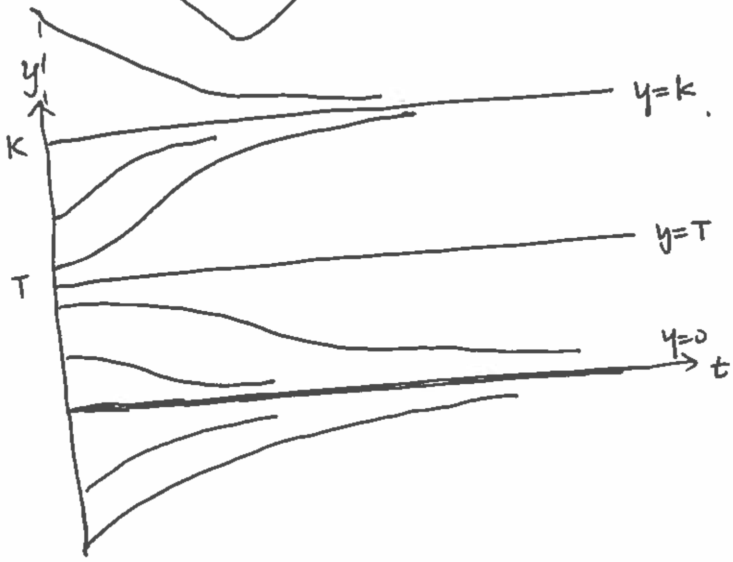
$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y. \qquad r>0 \quad 0<T<K.$$

Equilibrium solutions. $\boxed{y=0}$ stable, $\boxed{y=T}$ unstable, $\boxed{y=K}$ stable.

Phase line



Integral curves



If $y(0)=y_0 < T$ $\qquad \lim_{t\to\infty} y(t) = 0.$

If $y(0)=y_0 \in (T, K)$ $\qquad \lim_{t\to\infty} y(t) = K.$

If $y(0)=y_0 > K$ $\qquad \lim_{t\to\infty} y(t) = K.$

Section 2.6. Exact equations.

Ex! $\underbrace{2xy^2 + y}_{M(x,y)} + \underbrace{(2x^2y + x + \cos y)}_{N(x,y)} \frac{dy}{dx} = 0$

This is a 1st order nonlinear ODE, and it's not separable.

Suppose we are given a magical function $\psi(x,y) = x^2y^2 + xy + \sin y$.

Note that $\frac{\partial \psi}{\partial x} = 2xy^2 + y = M(x,y)$.

$\frac{\partial \psi}{\partial y} = 2x^2y + x + \cos y = N(x,y)$

Therefore, the DE can be written as.

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{d}{dx}\psi(x, y(x)) = 0 \Rightarrow \psi(x, y(x)) =$$

Recall chain rule $\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$

For Ex 1. $\frac{d}{dx}\psi(x,y) = 0$

$\frac{d}{dx}\left(x^2y^2 + xy + \sin y\right) = 0 \Rightarrow x^2y^2 + xy + \sin y = C$.

Implicit form of the general sol.

Def [Exact DE]

The ODE $M(x,y) + N(x,y)\,y' = 0$ is said to be exact if there is some function $\psi(x,y)$ s.t.

$$M = \frac{\partial \psi}{\partial x} \qquad N = \frac{\partial \psi}{\partial y}.$$

Then the sol. is $\psi(x,y) = C$.

Question: If we are not given a magical function ψ.

① How can we tell whether an eq. is exact?

② If it's exact, how do we find ψ?

Recall: equality of mixed derivatives.

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}.$$

For exact DE, there exists ψ st. $\frac{\partial \psi}{\partial x} = M$ $\frac{\partial \psi}{\partial y} = N$.

Then. $$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right).$$

$$\frac{\partial}{\partial y} M = \frac{\partial}{\partial x} N.$$

Exact DE satisfy $\boxed{\frac{\partial}{\partial y} M = \frac{\partial}{\partial x} N}$

Ex 1. $$\underbrace{2xy^2 + y}_{M(x,y)} + (\underbrace{2x^2y + x + \cos y}_{N(x,y)}) \frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = 4xy+1 \qquad \frac{\partial N}{\partial x} = 4xy+1$$ Equal.

Ex 2. $$(\underbrace{xy^2 + y}_{M(x,y)}) + (\underbrace{2x^2y + x}_{N(x,y)}) \frac{dy}{dx} = 0.$$

$$\frac{\partial M}{\partial y} = 2xy+1 \qquad \frac{\partial N}{\partial x} = 4xy+1$$ Not equal.

$\Rightarrow$ Not exact DE.

Theorem 2.6.1.

If. $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in the rectangular region

R: $\alpha < x < \beta$, $\gamma < y < \delta$. Then. the DE.

$$M(x,y) + N(x,y)\, y' = 0.$$

is an exact DE in R if and only if.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

at each point in R. That is, there exists. a function ψ. such that.

$\frac{\partial \psi}{\partial x} = M$. $\frac{\partial \psi}{\partial y} = N$. if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R

Ex: $2xy^2 + y + (2x^2y + x + \cos y)\frac{dy}{dx} = 0.$

Step 1. Use $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$ to test whether this DE is exact.

$\frac{\partial M}{\partial y} = 4xy + 1$

Yes.

$\frac{\partial N}{\partial x} = 4xy + 1$

Step 2. Want to find ψ s.t $\frac{\partial \psi}{\partial x} = M = 2xy^2 + y$

$\frac{\partial \psi}{\partial y} = N = 2x^2y + x + \cos y.$

Integrate M w.r.t x. where y is treated like a constant.

$\psi = \int M\,dx = \int (2xy^2 + y)\,dx$

$= x^2y^2 + xy + h(y).$

no matter what h(y) [is] it disappears when I take $\frac{\partial \psi}{\partial x}$.

Make sure $\frac{\partial \psi}{\partial y} = N$

Here $\frac{\partial \psi}{\partial y} = 2x^2y + x + h'(y) = 2x^2y + x + \cos$ [y]

$\Rightarrow h'(y) = \cos y.$

$\Rightarrow h(y) = \sin y + C.$

Plug h(y) into ψ. $\psi = x^2y^2 + xy + \sin y + C.$

Step 3: Write down the general sol.

$\psi(x,y) = x^2y^2 + xy + \sin y = C.$

Ex: $2y^2 + (6xy + 2)y' = 0.$

$\frac{\partial M}{\partial y} = 4y \quad \frac{\partial N}{\partial x} = 6y.$ Not exact.

But, after multiplying an integrating factor y to both sides

$2y^3 + (6xy^2 + 2y)y' = 0$ is exact.

$\frac{\partial M}{\partial y} = 6y^2 \quad \frac{\partial N}{\partial x} = 6y^2.$

However, there is no systematic way to find an integrating factor to make an DE exact.