Math 2552. Differential Equations Fall 2017

Today: an introductory example that has a bit of everything.

Example: $p(t)$ population of mice at time t.

Known: ① without predator, the rate of change of population is proportional to the current population. with growth rate r.

② owls eat k mice per day.

$$\frac{dp(t)}{dt} = r\,p(t) - k. \quad (\circledast)$$ 1st order ODE.

Some terms. t independent variable.

$p(t)$ dependent variable.

r, k. parameters that are constants.

ODE: only one independent variable.

PDE: more than one independent variables.

Order of DE: the highest order of derivatives.

eg. $y''(t) + (y'(t))^{10} = \sin t$. 2nd order ODE.

$\frac{\partial}{\partial t} p(t,x) + \frac{\partial}{\partial x} p(t,x) = 0$ 1st order. PDE.

Def [Solution] The solution of an differential equation is a function that satisfies the equation.

Note. It may be hard to solve a differential equation, but it is always easy to check whether a given function is a solution.

EX. Check that for every C, $p(t) = Ce^{rt} + \frac{k}{r}$ is a solution to $\circledast$

Let us plug $p(t)$ in $\circledast$.

$LHS = \frac{dp}{dt} = cre^{rt}$

$RHS = rp(t) - k = r(Ce^{rt} + \frac{k}{r}) - k = cre^{rt}$.

$LHS = RHS \Rightarrow$ Yes. $p(t)$ is a solution to $\circledast$

How to solve ⊛ ?

$$\frac{dp}{dt} = r\left(p(t) - \frac{k}{r}\right).$$

$$\frac{dp/dt}{p(t) - \frac{k}{r}} = r \quad \Rightarrow \int \frac{1}{p(t)-\frac{k}{r}}\, dp = \int r\, dt.$$

$$\Rightarrow \ln\left|p(t) - \frac{k}{r}\right| = rt + C. \quad \Rightarrow \left|p(t) - \frac{k}{r}\right| = e^{rt+C}.$$

$$\Rightarrow p(t) - \frac{k}{r} = \pm e^{C} e^{rt} \quad \Rightarrow p(t) = \underbrace{\boxed{\pm e^{C}}}_{\text{also a constant}} e^{rt} + \frac{k}{r}.$$

Let $C' = \pm e^{C}$ $\quad p(t) = C' e^{rt} + \frac{k}{r}$ is the general solution to ⊛

denote $C e^{rt} + \frac{k}{r}$.

Let us fix $k = 100$ $\quad r = 2$.

So $p(t) = C e^{2t} + 50$ is the general solution.

$\boxed{\text{Integral curves}}$ Each integral curve is associated with a particular value of C; it is the graph of the solution with C.

$\boxed{\text{Initial Value Problem}}$ IVP.

Sometimes $p(0)$ is given (called an initial condition) and finding a solution with this initial condition is called an <u>initial value problem</u>.

Ex: $p(0) = 600$. $\quad p(0) = C + 50 = 600 \quad C = 550.$

So $p(t) = 550 e^{2t} + 50$ is the solution to the IVP.

<u>Qualitative methods.</u>

Def. A 1st order autonomous DE is an equation of the form.

$$\frac{dy}{dt} = f(y).$$

Ex: Are the following DE autonomous?

$p'(t) = rp - k$ ✓

$x'(t) + 2x = (\sin x)^2$ ✓.

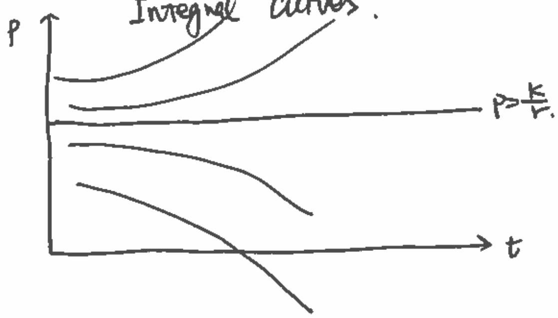
$x'(t) = x + e^{t}$ ✗.

For an autonomous eq. any constant function $y = c$ must satisfy $f(c) = 0$.
These are called <u>equalibrium solutions</u>.

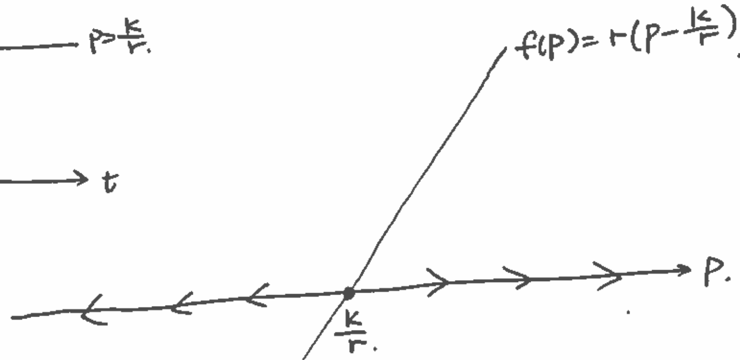
Ex. for $\frac{dp}{dt} = rp - k$.

Equilibrium: $rp = k$ $\quad p = \frac{k}{r}$.

Integral curves.



<u>Phase line</u> (1 dim)



when $p < \frac{k}{r}$ $\quad \frac{dp}{dt} < 0$. $\quad P \downarrow$.

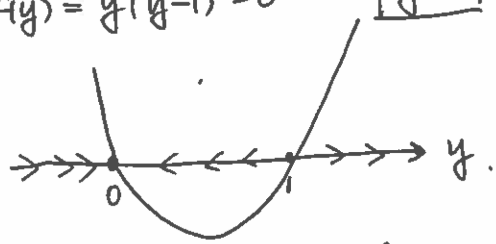
when $p > \frac{k}{r}$ $\quad \frac{dp}{dt} > 0$ $\quad P \uparrow$

The phase line can be used to sketch the integral curves.

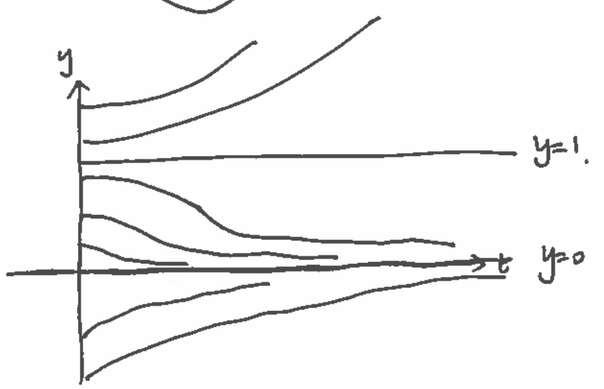
Ex. $\frac{dy}{dt} = f(y) = y(y-1)$.

Equilibrium: $f(y) = y(y-1) = 0$ $\quad \boxed{y=0}$ & $\boxed{y=1}$

Phase line



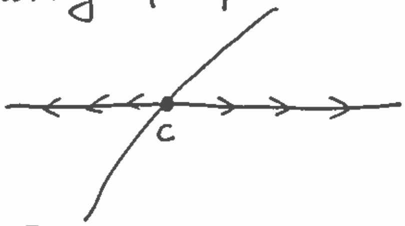
Integral curves.



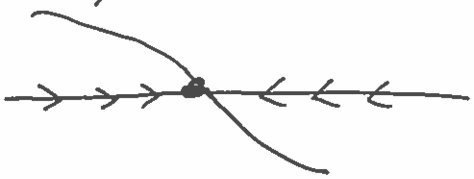
$y = 1$: unstable

$y = 0$: stable.

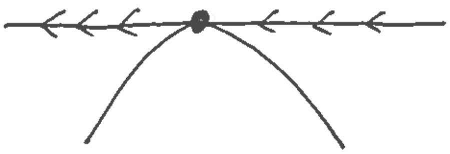
Stability of equilibrium. $f(c)=0$.



unstable



stable.



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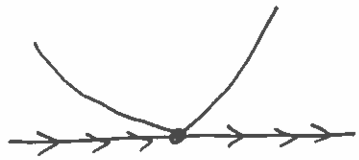


semistable.

! See Textbook Table 1.2.1

Direction fields.

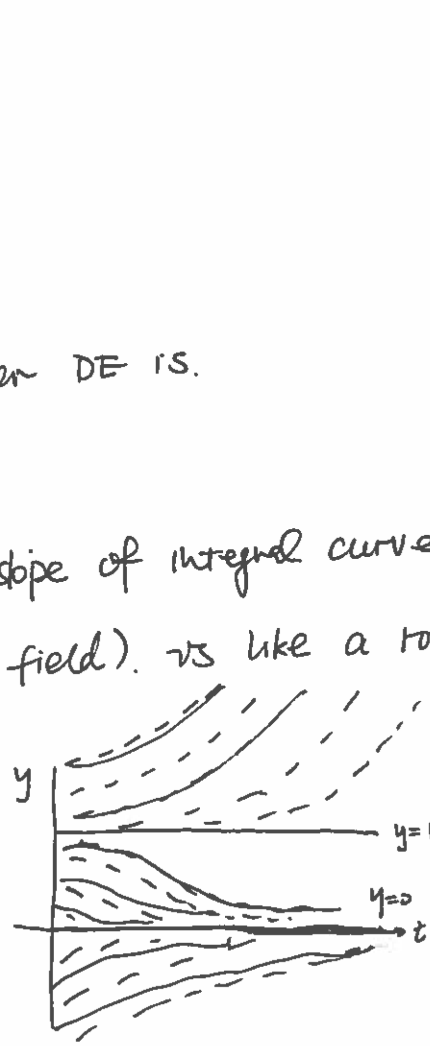
The standard form of a 1st order DE is.

$$\frac{dy}{dt} = f(t,y).$$

So, at the point (t,y), the slope of integral curve should be $f(t,y)$.

Direction filed (also called slope field) is like a road map for solution.

Ex: $\frac{dy}{dt} = \underbrace{y(y-1)}_{\text{autonomous eq.}}$



google "slope field generator."

More terms:

ODE & PDE.

If the unknow function depends on only 1 independent variable $\rightarrow$ ODE

more than 1 variable $\rightarrow$ PDE.

eg: $f''(t) + f(t) - \sin t = 0.$ 2nd order ODE.

$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$ 2nd order PDE.

Linear & nonlinear. DE.

An n-th order. ODE can be written as

$$F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0.$$

If F is linear in $y, y', \ldots y^{(n)}$, then it is called a linear. DE.

otherwise, it is nonlinear.

That is.. an n-th order ODE is of the form.

$$a_0(t) y + a_1(t) y^{(1)} + a_2(t) y^{(2)} + \cdots + a_n(t) y^n = g(t).$$

For a linear DE, if $g(t) = 0$, $\Rightarrow$ homogeneous.

otherwise $\Rightarrow$ inhomogeneous or non-homogeneous.

eg: $y'' - \sin(t) y = e^t$ 2nd order linear. inhomogenous, ODE.

$y' = \sqrt{1+y^2} + t$. 1st order nolinear. ODE.

$t y''' + t^2 y' - e^t y = 0.$ 3rd order. Linear homogenous ODE.

The general soluton of nth order ODE has n undetermined constants.

so the initial value problem. is often of the form.

$y(t_0) = y_0$ $y'(t_0) = y_1$ .... $y^{(n-1)}(t_0) = y_{n-1}$