Midterm2 – 2552, Fall 2018

Instructor: Wenjing Liao

- Test time: 50 minutes.
- Please do not assist another person in the completion of this exam. Please do not copy answers from another student's exam. Please do not have another student take your exam for you. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted.
- No books or calculators are allowed.
- Only one sheet of handwritten note (front and back, letter size) is allowed.
- Read each problem carefully. Show all work for full credit.
- Make sure you have **10 pages**, including the cover page (Page 1), the page of scores (Page 2), and two blank pages (Page 9 and 10).

Your name: ______________

Circle your TA's name

Weiwei Zhang   Swagath Saraogi   Benjamin Ide   John Chiles

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(2)	
(3)	
(4)	
Total	

Problem 1 (10 points): For the following inhomogeneous equations, write down a suitable form for the particular solution $Y(t)$ if the method of underdetermined coefficients is to be used. Do not solve for the coefficients.

(1) $y'' + y' = t + t^2 e^{-t} + e^{-t}\cos t$. (5 points)

Characteristic equation. $\lambda^2 + \lambda = 0$ $\quad \lambda(\lambda+1) = 0$.

Eigenvalues. $\lambda_1 = 0$ $\quad \lambda_2 = -1$.

Term t : $Y_1(t) = t(A_1 t + A_2)$ $\quad s=1$ since 0 appears once in the roots.

Term $t^2 e^{-t}$: $Y_2(t) = te^{-t}(B_1 t^2 + B_2 t + B_3)$. $\quad s=1$ since -1 appears once in the roots.

Term $e^{-t}\cos t$: $Y_3(t) = e^{-t}(C\cos t + D\sin t)$.

$Y(t) = t(A_1 t + A_2) + te^{-t}(B_1 t^2 + B_2 t + B_3) + e^{-t}(C\cos t + D\sin t)$.

(2) $y'' + 2y' + 2y = t + e^{-t} + e^{-t} t \sin t$. (5 points)

Characteristic equation: $\lambda^2 + 2\lambda + 2 = 0$. $\quad \lambda = \frac{-2 \pm \sqrt{2^2 - 4\cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$

$= \frac{-2 \pm 2i}{2} = -1 \pm i$

Eigenvalues. $\lambda_1 = -1 + i$ $\quad \lambda_2 = -1 - i$

Term t : $Y_1(t) = A_1 t + A_2$.

Term e^{-t} : $Y_2(t) = Be^{-t}$.

Term $e^{-t} t \sin t$ $\quad Y_3(t) = t\left[e^{-t}(C_1 t + C_2)\sin t + e^{-t}(D_1 t + D_2)\cos t\right]$.

$Y(t) = A_1 t + A_2 + Be^{-t} + te^{-t}(C_1 t + C_2)\sin t + te^{-t}(D_1 t + D_2)\cos t$.

Problem 2: (10 points): (1) Find two fundamental solutions of the homogeneous equation $y'' + y = 0$. (4 points)

Characteristic eq: $\lambda^2 + 1 = 0 \qquad \lambda^2 = -1$

$\lambda_1 = i \qquad \lambda_2 = -i.$

$y_1(t) = \cos t \qquad y_2(t) = \sin t.$

(2) Find a particular solution of the inhomogeneous equation

$$y'' + y = \frac{1}{\sin t}.$$

(6 points)

$$W[y_1\ y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$y_1(t) = \cos t \qquad y_2(t) = \sin t \qquad g(t) = \frac{1}{\sin t}$

$$Y(t) = -y_1(t) \int \frac{y_2(t)\, g(t)}{W[y_1, y_2](t)}\, dt + y_2(t) \int \frac{y_1(t)\, g(t)}{W[y_1, y_2](t)}\, dt$$

$$= -\cos t \int (\sin t) \frac{1}{\sin t}\, dt + \sin t \int (\cos t) \frac{1}{\sin t}\, dt.$$

$$= -\cos t \int 1\, dt + \sin t \int \frac{\cos t}{\sin t}\, dt.$$

$$= -t + (\sin t) \ln|\sin t|.$$

$u = \sin t. \quad du = \cos t\, du$

$$\int \frac{\cos t}{\sin t}\, dt = \int \frac{du}{u} = \ln|u| = \ln|\sin t|$$

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Problem 3 (10 points): Consider the equation

$$y'' - y' - 2y = 0.$$

(1) Convert this second order equation to the first order system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

by letting $x_1 = y$ and $x_2 = y'$. (3 points)

$x_1' = y' = x_2$

$x_2' = y'' = y' + 2y = 2x_1 + x_2$.

$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$.

(2) Find the fundamental matrix e^{At} for the matrix A in Part (1). (7 points)

$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{pmatrix}$.

$\det(A-\lambda I) = -\lambda(1-\lambda) - 2 = \lambda^2 - \lambda - 2$
$= (\lambda-2)(\lambda+1)$.

Eigenvalues: $\lambda_1 = 2$ $\lambda_2 = -1$

Eigenvectors: $\lambda_1 = 2$ $A - \lambda I = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$

$-2v_1 + v_2 = 0$ $v_2 = 2v_1$

$\vec{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -1$ $A - \lambda I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

$v_1 + v_2 = 0$.

$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Fudamtal sol. $\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2 = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$X(t) = \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}$.

$X(0) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$.

Compute $X^{-1}(0)$: $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$ $X^{-1}(0) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$.

$e^{At} = X(t)X^{-1}(0) = \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} = \frac{1}{3}\left(\begin{array}{c|c} e^{2t}+2e^{-t} & e^{2t}-e^{-t} \\ 2e^{2t}-2e^{-t} & 2e^{2t}+e^{-t} \end{array}\right)$

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Problem 4 (10 points): Find the general solution of

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Notice that $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda+1)^3$, but you need to show your work about the computation of $\det(A - \lambda I)$.

$$A-\lambda I = \begin{pmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{pmatrix} \leftarrow \text{expand along this row.}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1 & 2 \\ -3-\lambda & -7 \end{vmatrix} - \lambda \begin{vmatrix} -\lambda & 1 \\ -5 & -3-\lambda \end{vmatrix}$$

$$= -7 - 2(-3-\lambda) - \lambda\left(-\lambda(-3-\lambda)+5\right)$$

$$= -7+6+2\lambda - \lambda(\lambda^2+3\lambda+5)$$

$$= -1+2\lambda-\lambda^3-3\lambda^2-5\lambda$$

$$= -\lambda^3-3\lambda^2-3\lambda-1$$

$$= -(\lambda+1)^3$$

$\lambda = -1$ algebraic multiplicity $m=3$.

$$A-\lambda I = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$(A-\lambda I)^2 = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$(A-\lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solve $(A-\lambda I)^3 \vec{v} = \vec{0}$ gives.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\vec{x}_1(t) = e^{-t}\left[\vec{v}_1 + t(A-\lambda I)\vec{v}_1 + \frac{t^2}{2}(A-\lambda I)^2\vec{v}_1\right]$$

$$= e^{-t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ -2 & -1 & -3 \end{pmatrix}\right]$$

$$= e^{-t}\begin{pmatrix} 1+t-t^2 \\ -5t-t^2 \\ t-t^2 \end{pmatrix}.$$

$$\vec{x}_2(t) = e^{-t}\left[\vec{v}_2 + t(A-\lambda I)\vec{v}_2 + \frac{t^2}{2}(A-\lambda I)^2\vec{v}_2\right]$$

$$= e^{-t}\begin{pmatrix} t-\frac{1}{2}t^2 \\ 1-2t-\frac{1}{2}t^2 \\ -\frac{1}{2}t^2 \end{pmatrix}$$

$$\vec{x}_3(t) = e^{-t}\left[\vec{v}_3 + t(A-\lambda I)\vec{v}_3 + \frac{t^2}{2}(A-\lambda I)^2\vec{v}_3\right]$$

$$= e^{-t}\begin{pmatrix} 2t-\frac{3}{2}t^2 \\ -7t-\frac{3}{2}t^2 \\ 1+t-\frac{3}{2}t^2 \end{pmatrix}.$$

General sol.

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$