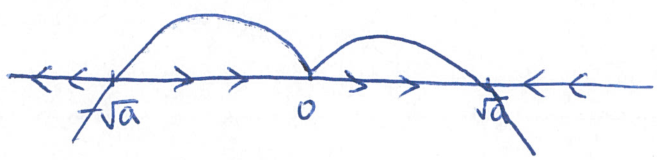
Problem 1 (10 points): Consider the equation

$$\frac{dy}{dt} = y^2(a - y^2).$$

(1) When $a > 0$, find all equilibrium solutions, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable? (4 points)

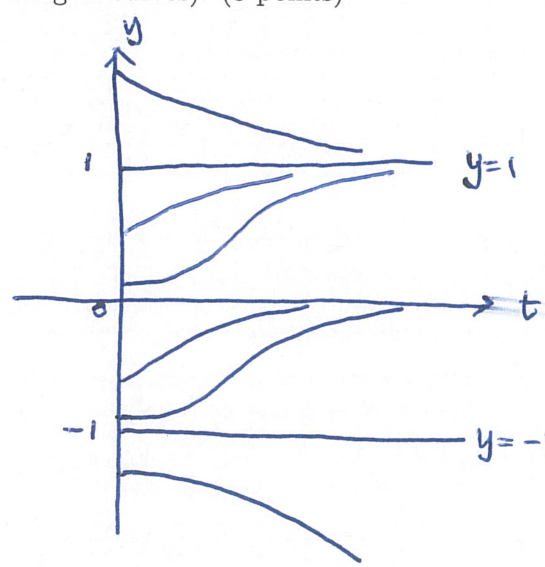
Equilibrium $y=0$ $y=\sqrt{a}$ $y=-\sqrt{a}$



$-\sqrt{a}$ unstable 0 semistable.

$\sqrt{a}$ stable

(2) When $a = 1$, sketch several solutions on the ty plane (same as integral curves). (3 points)



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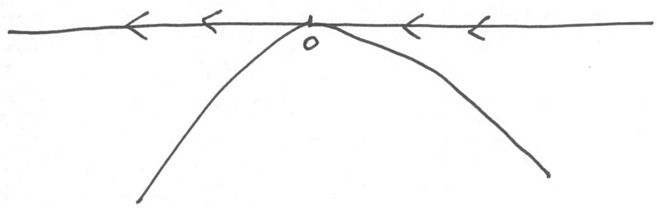
(3) When $a < 0$, find all equilibrium solutions, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable? (3 points)

When $a<0$, $a-y^2<0$

Equilibrium $y=0$



0 semistable.

Problem 2: (10 points): Consider the following equation

$$(ye^{xy}\sin x + e^{xy}\cos x + 2x)dx + xe^{xy}\sin x dy = 0.$$

(1) Show this equation is exact. (4 points)

$M(x,y) = ye^{xy}\sin x + e^{xy}\cos x + 2x$

$N(x,y) = xe^{xy}\sin x$

$$\frac{\partial M}{\partial y} = e^{xy}\sin x + xye^{xy}\sin x + xe^{xy}\cos x + 0.$$

$$\frac{\partial N}{\partial x} = \frac{\partial x}{\partial x}e^{xy}\sin x + x\frac{\partial}{\partial x}(e^{xy})\sin x + xe^{xy}\frac{\partial}{\partial x}(\sin x)$$

$$= e^{xy}\sin x + xye^{xy}\sin x + xe^{xy}\cos x$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so this equation is exact.

(2) Find the general solution. No need to write the solution in an explicit form. (6 points)

$\Phi(x,y)$ satisfies. $\frac{\partial \Phi}{\partial x} = M$ $\frac{\partial \Phi}{\partial y} = N.$

$$\Phi = \int N dy = \int xe^{xy}\sin x\, dy = e^{xy}\sin x + h(x).$$

$$\frac{\partial \Phi}{\partial x} = ye^{xy}\sin x + e^{xy}\cos x + h'(x)$$

Set $\frac{\partial \Phi}{\partial x} = M$ then $h'(x) = 2x \Rightarrow h(x) = x^2.$

General sol:

$$\Phi(x,y) = e^{xy}\sin x + x^2 = C.$$

Problem 3 (10 points): Solve the initial value problem
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$
$$x(0) = 1,\ y(0) = 1.$$

$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$

$\det(A-\lambda I) = -\lambda(2-\lambda)-(-1)$
$= \lambda^2 - 2\lambda + 1$
$= (\lambda-1)^2.$

$\lambda_1 = 1 \qquad \lambda_2 = 1$

Eigenvector:

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 + v_2 = 0.$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

the first fundamental sol is.

$\vec{x}_1(t) = e^{\lambda t}\vec{v} = e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The second one is $\vec{x}_2(t) = e^{\lambda t}(t\vec{v} + \vec{w}).$

where $(A - \lambda I)\vec{w} = \vec{v}$

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

$w_1 + w_2 = 1$

Take $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_2(t) = e^t\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$

General sol.

$\vec{x}(t) = c_1 e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^t\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$

Plug in initial value. $t=0$ $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$c_1 = 1$
$-c_1 + c_2 = 1 \Rightarrow c_2 = 2.$

Sol.

$\vec{x}(t) = e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2e^t\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$

Problem 4 (10 points): Consider the following inhomogeneous system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -7 \\ 1 \end{bmatrix}.$$

(1) What is the equilibrium solution? (2 points)

Set $\begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$2x + 5y = 7 \quad \Rightarrow y = 1$$
$$-x \quad = -1 \quad \Rightarrow x = 1$$

$$\vec{x}_{eq} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(2) Find the general solution of the homogeneous system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

(5 points)

$A = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 5 \\ 1 & -\lambda \end{bmatrix}$

$$\det(A - \lambda I) = -\lambda(2-\lambda) - 5 = \lambda^2 - 2\lambda - 5$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$
$$= 1 \pm 2i$$

$\lambda_1 = 1 + 2i \qquad \begin{bmatrix} 2-(1+2i) & 5 \\ 1 & -(1+2i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1-2i & 5 \\ 1 & -1-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-2i)v_1 + 5v_2 = 0$$

$$\vec{v} = \begin{bmatrix} -5 \\ 1-2i \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\vec{x}_1(t) = e^t\left(\begin{bmatrix} -5 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)$$

$$\vec{x}_2(t) = e^t\left(\begin{bmatrix} -5 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t\right).$$

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

(3) Write down the general solution of the inhomogeneous system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -7 \\ 1 \end{bmatrix}.$$

(1 points)

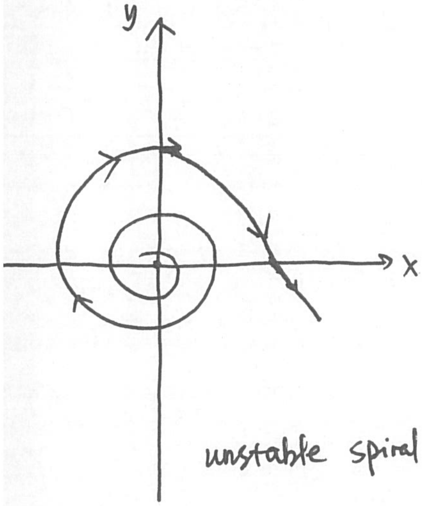
$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_{eq}.$$

$$= c_1 e^t\left(\begin{bmatrix} -5 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)$$

$$+ c_2 e^t\left(\begin{bmatrix} -5 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t\right) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

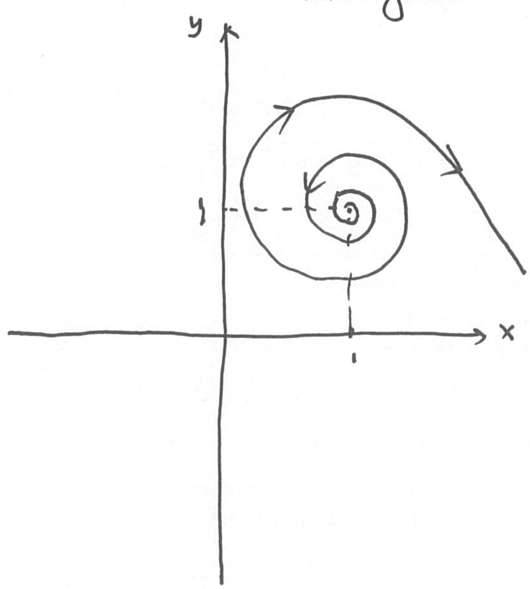
(4) Draw a phase portrait for the inhomogeneous system. (2 points)

Phase portrait for the homogeneous system



unstable spiral.

inhomogeneous system.



$$\frac{dx}{dt} = 2x + 5y$$

$$\frac{dy}{dt} = -x$$

At $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \frac{dx}{dt} = 5 \quad \frac{dy}{dt} = 0.$

At $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \frac{dx}{dt} = 2 \quad \frac{dy}{dt} = -1$

Problem 5 (10 points): Consider the equation

$$\frac{dy}{dt} = y^{\frac{1}{2}}, \qquad y(0) = 0.$$

(1) Find all solutions of this equation. (6 points)

First $y=0$ is a solution.

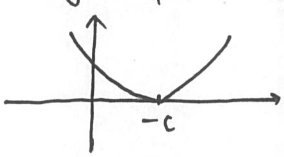
$\int y^{-\frac{1}{2}} dy = \int dt$.

$\frac{1}{1-\frac{1}{2}} y^{\frac{1}{2}} = t + C$.

$2y^{\frac{1}{2}} = (t+C)$.

$y^{\frac{1}{2}} = \frac{1}{2}(t+C)$

$y = \frac{1}{4}(t+C)^2$.



All possible general solutions.

① $y=0$ ② $y=\frac{1}{4}(t+C)^2$

③ $y=\begin{cases} 0 & t \le -C \\ \frac{1}{4}(t+C)^2 & t > -C \end{cases}$ ④ $y=\begin{cases} \frac{1}{4}(t+C)^2 & t \le -C \\ 0 & t > -C \end{cases}$

Need $(y(0)=0$.

① $y=0$ ② $C=0$ $y=\frac{1}{4}t^2$.

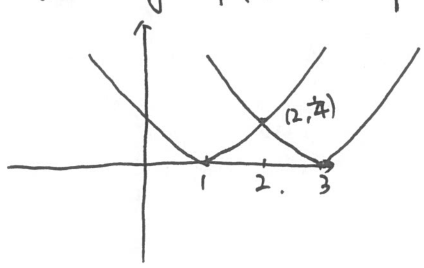
③ $y=\begin{cases} 0 & t \le -C. \\ \frac{1}{4}(t+C)^2 & t > -C \end{cases}$ $-C \ge 0 \Rightarrow C \le 0$.

④ $y=\begin{cases} \frac{1}{4}(t+C)^2 & t \le -C \\ 0 & t > -C \end{cases}$ $-C \le 0 \Rightarrow C \ge 0$.

(2) Is there a solution that passes through the point $(2, \frac{1}{4})$? If so, find all solutions that pass through this point. (4 points)

Let. $\frac{1}{4}(2+C)^2 = \frac{1}{4}$ $(C+2)^2 = 1$ $C+2 = \pm 1$ $C=-1$ or $C=-3$.

$y=\frac{1}{4}(t-1)^2$ and $y=\frac{1}{4}(t-3)^2$ pass through $(2, \frac{1}{4})$.



Need to satisfy $y(0)=0$ as well, so the only solution is.

$$y=\begin{cases} 0 & t \le 1 \\ \frac{1}{4}(t-1)^2 & t > 1. \end{cases}$$